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Rigid D6-branes on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ with discrete torsion

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Abstract

We give a complete classification of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ orientifolds on factorisable tori and rigid D6-branes on them. The analysis includes the supersymmetry, RR tadpole cancellation and K-theory conditions and complete massless open and closed string spectrum (i.e. non-chiral as well as chiral) for fractional or rigid D6-branes for all inequivalent compactification lattices, without and with discrete torsion. We give examples for each orbifold background, which show that on $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_6'$ there exist completely rigid D6-branes despite the self-intersections of orbifold image cycles. This opens up a new avenue for improved Standard Model building. On the other hand, we show that Standard and GUT model building on the $\mathbb{Z}_2 \times \mathbb{Z}_4$ background is ruled out by simple arguments.

Contents

1 Introduction

To date the Standard Model of particle physics (SM) is in very good agreement with experiment. Still, there are several theoretical reasons to believe that it is an effective theory valid only up to some scale[1]. The SM has about 17 parameters determined only by fitting with data. In an underlying more fundamental theory these parameters might be related via symmetry and/or determined dynamically due to expectation values of extra SM singlets. In particular the electro-weak symmetry breaking scale is not explained within the SM. Therefore, it is natural to believe that the SM is an effective description of nature up to that scale. The, perhaps, most prominent theory beyond the SM (BSM) is low energy supersymmetry. It explains neatly many open questions of the SM (for reviews see e.g. [1–3]). In the supersymmetric extension of the SM some parameters are indeed related by symmetry, e.g. the quartic Higgs coupling is expressed in terms of the $SU(2)_L$ and $U(1)_Y$ gauge couplings. However, it is not known which mechanism of supersymmetry breaking is realised in nature. Pragmatically, all possible schemes are parameterised by the so called soft breaking terms introducing 105 new parameters. So, the number of parameters actually grows but just because details of the BSM are not known yet. Conceptually the number is reduced by choosing a particular breaking scheme.

So far, our discussion focused on the so called bottom up approach: The BSM consists of the SM plus extensions. In a top down approach, on the other hand, one starts with more fundamental questions, for instance: How gravity is quantised? In an ideal world the answer to such a question would ultimately lead to the SM with the correct values for its 17 parameters. Indeed, string theory consistently includes quantum gravity and has only one parameter, the string tension $1/\alpha'$. However, there is a huge landscape of consistent string vacua and no conceptual way of choosing one is known. Parameters are indeed replaced by expectation values of moduli. There are, however, many ways to stabilise these moduli. Again, one can postpone the question of vacuum selection and pragmatically look for string vacua reproducing correctly the SM at low energies. There are several approaches to identify promising string vacua each of which is well motivated, see e.g. [4–7] for heterotic orbifolds, [8, 9] for heterotic Calabi-Yau compactifications with $SU(N)$ bundles and [10, 11] with $U(N)$ bundles, [12–15] for local IIB models, [16–18] for F-theory and [19, 20] for Gepner models. While the techniques for identifying the gauge group and chiral matter spectrum are rather straightforward in all approaches, an investigation of the exact field theory is typically constrained to vacua, where e.g. conformal field theory methods can be used.

In the present paper, we take intersecting D6-branes of type IIA string theory as our

[1] There are also some experimental and observational hints such as e.g. dark matter.

starting point. There are several good reviews on this kind of model building containing also references to the original literature (including the T-dual constructions of magnetised branes in IIB string theory), e.g. [21–27]. Geometrically, intersecting D6-brane constructions are quite intuitive. Closed strings move through the bulk. Their excitations provide the gravitational sector of the low energy theory. Open strings ending on the same stack of D6-branes give rise to the gauge sector, while strings stretched between different stacks of D6-branes provide potentially chiral matter. Type IIA string theory will be compactified on an orbifold of T^6 such that the amount of supersymmetry is reduced to $\mathcal{N} = 2$. This simple geometry of the compact space has the advantage that conformal field theory techniques can be applied. Explicit computations of the complete spectrum, interaction terms and instanton corrections are possible. Gauging further an orientifold symmetry reduces the amount of supersymmetry to $\mathcal{N} = 1$. In type IIA, orientifold symmetries include a reflection of an odd number of directions. We choose this number equal to three and add thus orientifold-six-planes (O6-planes). RR-charges are finally cancelled by adding D6-branes. Throughout this article, we focus on *"globally consistent string compactifications"* in the sense of cancellation of *all* untwisted and twisted RR tadpoles and fulfillment of the K-theory constraint, where we also know the *full* closed and open string spectrum and can in principle compute the exact moduli dependence of couplings to all orders. This is in contrast to a recent trend of calling *"locally consistent"* or anomaly-free gauge quivers "globally consistent", see e.g. [28, 29].

In the present paper, we will take $\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ with $2M = 2, 4, 6$ and $6'$ as orbifold group. These belong to a set for which discrete torsion can be turned on [30, 31]. There the authors consider $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds and discuss possible phase factors in front of twisted sector contributions to the torus amplitude. Non-trivial phase factors correspond to discrete torsion. For our subset discrete torsion is related to a non-trivial second root of unity, i.e. a sign choice. Our motivation is that with discrete torsion one can have *rigid D6-branes*. These are D6-branes wrapping a fractional bulk three-cycle plus an exceptional three-cycle which is collapsed to a lower dimensional fixed cycle in the orbifold limit of the Calabi-Yau space [32]. Since D6-branes wrapping such cycles cannot change position (at least in some direction) the corresponding adjoint moduli are absent from the open string spectrum. (However, for $2M > 2$ adjoints can "reappear through the back-door" from open strings stretching to intersecting orbifold images [33, 34]. In that case adjoint moduli are associated with brane recombination. Whether these moduli are really present has to be seen in a case by case study.) In the absence of adjoint matter, the arbitraryness of breaking the gauge group along a flat direction is removed, and the values of beta function coefficients are improved in view of phenomenology. Rigid branes also admit non-vanishing instanton contributions to the superpotential, Kähler potential [35, 36] and gauge kinetic function due to their minimal number of zero modes (see e.g. the review [37] and references therein),

which can generate non-perturbative neutrino masses [38, 39], μ-terms [40], perturvatively forbidden Yukawa couplings [41, 42] or might even trigger supersymmetry breaking [43].

Orbifolds with discrete torsion are also of phenomenological interest, since they have a reduced number of twisted moduli compared to their counterparts without torsion. This is due to the fact that discrete torsion does not only exchange Kähler for complex structure moduli in some twist sectors, but also projects out states at points fixed under all orbifold generators.

While twisted moduli are frozen at the orbifold point and untwisted complex structure moduli are stabilised by the supersymmetry conditions on D6-branes, the stabilisation of the dilaton and untwisted Kähler moduli requires the introduction of closed string background fluxes. The impact on D6-brane model building has to date been mainly discussed on the torus background, see e.g. [44] with some partial first results for orbifolds in [45–47]. A non-trivial H-flux is closely connected to Scherk-Schwarz compactifications (see e.g. [48] and references therein) and freeley-acting orbifolds (e.g. [49]).

In [50, 51, 33] the case $2M = 2$ ($\mathbb{Z}_2 \times \mathbb{Z}_2$) was studied and toy models with removed adjoint moduli were presented.[2] Another, empirical, observation favouring the discrete torsion orbifolds has been made in [57]: In cases where there are no three family models without discrete torsion (*viz.* $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the AAA lattice [53, 54] or on non-factorisable T^6-orbifolds [58]) it has been demonstrated that with discrete torsion and rigid D6-branes there are three family models.

The other orbifolds for $2M \in \{4, 6, 6'\}$ with discrete torsion have to our knowledge not been studied before in view of their potential for D6-brane model building. Note in particular, that for $2M > 2$, the IIA models on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$, both without and with discrete torsion, in this article are not T-dual to any of the IIB models on $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$, see e.g. [59–63], since T-duality maps a symmetric orbifold to an asymmetric one.

The first case $2M = 4$ is closely related to the same orbifold background without torsion, which has been studied before in [64–66], and for which a no-go theorem for three supersymmetric SM generations exists [67]. The second case $2M = 6$ has $T^6/\mathbb{Z}_6'$ as a subsector, which has proven to be able to accommodate the SM gauge group and chiral spectrum [68, 69], however, always with some adjoint matter. The last case $2M = 6'$ has $T^6/\mathbb{Z}_6$ as a subsector, for which also three generation models with some adjoint matter content are known [70, 71].

[2]For D6-branes on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold without discrete torsion and three adjoints per stack, see e.g. the first non-chiral [52] and chiral models in [53, 54], a statistical treatment in [55, 56] and the further references in the review articles [21–26].

In the context of fractional D6-branes on $T^6/\mathbb{Z}_{2N}$ backgrounds, a method of deriving the complete matter spectrum from the beta function coefficients, which are computed along with the gauge thresholds,[3] has been derived [79], on which we will heavily rely after appropriate modifications to $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$. This method will in particular be used to confirm the absence of adjoint matter, i.e. the complete rigidity of the D6-branes, for explicit examples.

Outline

This paper is organised as follows: in section 2, we discuss the general set-up of D6-branes on orbifolds with discrete torsion. This includes the various twist sectors, three cycles, supersymmetry and RR tadpole cancellation and K-theory conditions and chiral spectrum, as well as exotic O6-planes.

In section 3, we review the known $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$ example and generalise it to arbitrary backgrounds with tilted tori. The K-theory constraints for the case with tilted tori are discussed in great detail, since the argumentation carries over to the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ backgrounds. The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold without and with discrete torsion is presented in section 4, and the no-go theorem on three generation models is generalised from the known case without torsion to the one with discrete torsion. The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ cases are discussed in sections 5 and 6, respectively, where the full lattices of three-cycles are worked out. We give some globaly consistent, i.e. RR tadpole cancelling and K-theory constraint fulfilling, supersymmetric example for each orbifold background at the end of the corresponding section, for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ only in the presence of discrete torsion, but for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ for both choices of torsion.

Sections 3 to 6 can basically be read independently of each other.

Section 7 contains our conclusions and outlook. Finally, in appendices A to D, we collect technical details on the computation of the complete massless open and closed string spectrum, a complete list of assignments of exceptional three-cycles to a given bulk three-cycle for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, and last but not least we present the relations of product orbifolds $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ without and with discrete torsion to orbifolds with only one generator, $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$.

[3] For earlier computations of gauge thresholds for intersecting D6-branes, see [72, 73] on the torus background and [74] on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ background with discrete torsion and [75] for the dual description with D9-branes in IIB. For gauge thresholds in local models see [76–78].

2 IIA string theory with D6-branes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbi- and orientifolds without and with torsion

2.1 Generalities on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds

2.1.1 Geometric set-up

For the six extra dimensions of type IIA string theory we choose complex coordinates z_1, z_2, z_3 with $z_k \equiv x_{2k-1} + i\, x_{2k}$. These are compactified on a factorisable six-torus $T^6 = (T^2)^3$. This T^6 is constructed by dividing each of the complex planes by a two dimensional lattice to be specified shortly. From T^6 we obtain an orbifold via identification of points related by a discrete subgroup of $SU(3)$ which is our orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_{2M}$. Its generators θ and ω, respectively, act on the coordinates as

$$\theta : \quad z_i \to \mathrm{e}^{2\pi \mathrm{i} v_i}\, z_i\,, \qquad \omega : \quad z_i \to \mathrm{e}^{2\pi \mathrm{i} w_i}\, z_i, \tag{1}$$

with $\sum_{i=1}^{3} v_i = \sum_{i=1}^{3} w_i = 0$. The v_i's and w_i's are integer multiples of $1/2$ and $1/2M$, respectively. For the list of orbifolds to be considered, they are explicitly taken as

$$\vec{v} = \frac{1}{2}\,(1,-1,0)\,, \qquad \begin{cases} \vec{w} = \frac{1}{2M}\,(0,1,-1) \quad \text{with} \quad 2M \in \{2,4,6\} \\ \vec{w} = \frac{1}{6}\,(-2,1,1) \quad \text{for } \mathbb{Z}_2 \times \mathbb{Z}_6' \end{cases} . \tag{2}$$

Finally, we obtain an orientifold by identifying strings related by $\Omega\mathcal{R}$ where Ω changes the orientation of the string worldsheet and $\mathcal{R}$ acts as complex conjugation on the z_i. The lattices have to be taken such that $\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ as well as $\mathcal{R}$ are lattice automorphisms.

For any two dimensional lattice, $\mathbb{Z}_2$ is always an automorphism. Non-trivial restrictions come from imposing $\mathbb{Z}_3$ or $\mathbb{Z}_4$, and $\mathcal{R}$ invariance. Figure 1 displays the two choices imposed by $\mathcal{R}$ invariance, whereas figure 2 shows $\mathbb{Z}_4$ and $\mathbb{Z}_3$ invariant lattices (the latter one is also $\mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$ invariant). Imposing $\mathbb{Z}_4$ or $\mathbb{Z}_3$ invariance on the $\mathcal{R}$ invariant lattices fixes the ratio of the lengths of the lattice basis, or radii of the corresponding one-cycles as explained in the caption of figure 1. Alternatively, one can impose $\mathcal{R}$ invariance on the orbifold invariant lattice. This fixes the orientation of the real axis (x_{2i-1}) within the plane leaving the green or yellow coordinate systems of the z_i in figure 2.

Of particular importance are fixed points under the orbifold group. States of twisted sector closed strings are localised at such points. More importantly for us, also exceptional cycles to be discussed below are situated at fixed points. Figures 1 and 2 show the location of

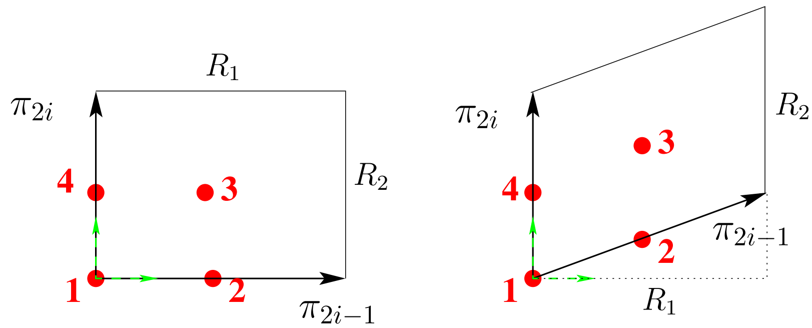

Figure 1: The $\mathbb{Z}_2$ invariant **a**-type (left) and **b**-type (right) lattices. The $\mathcal{R}$ invariant x_{2i-1} axis is along the 1-cycle $\pi_{2i-1} - b\,\pi_{2i}$ with $b = 0$ for the **a**-type lattice and $b = \frac{1}{2}$ for the **b**-type lattice. $\mathcal{R}$ acts as reflection along the x_{2i} axis, which is spanned by the 1-cycle π_{2i}. $\mathbb{Z}_2$ fixed points are depicted in red. For the **a**-type lattice, all $\mathbb{Z}_2$ fixed points are invariant under $\mathcal{R}$, whereas for the **b**-type lattice, only 1 and 4 are invariant while $2 \overset{\mathcal{R}}{\leftrightarrow} 3$. The $\mathbb{Z}_4$ and $\mathbb{Z}_6$ invariant lattices displayed in figure 2 can be mapped onto these two lattices for specific values of the radii as follows: the **A**-type $\mathbb{Z}_4$ invariant lattice is obtained by setting $R_1 = R_2$ on the $\mathbb{Z}_2$ invariant **a**-type lattice. The **b**-type lattice with $R_2/R_1 = 2, 2\sqrt{3}, 2/\sqrt{3}$ is a reparameterisation of the $\mathbb{Z}_4$ invariant **B**-type lattice and the $\mathbb{Z}_6$ invariant **A**- and **B**-type lattices, respectively.

a fixed point (or torus) within one complex direction. More details are discussed in the captions.

Orientifold planes extend along the non-compact directions and an $\mathcal{R}$ invariant three cycle on the orbifold (on T^6 this corresponds to an $\mathcal{R}$ invariant cycle plus its orbifold images). To cancel RR charges of O6-planes, D6-branes are added. They as well extend along the non-compact directions and wrap three-cycles on the underlying T^6. Bulk cycles correspond to such a three-cycle plus all its orbifold images. Further, there are exceptional and fractional cycles stuck at the orbifold fixed points to be discussed in section 2.2.

2.1.2 Discrete Torsion and Exotic O-Planes

Here, we consider string theory compactified on an orbifold. Imposing modular invariance on torus amplitudes shows the importance of twisted sector contributions. This can be easily seen as follows. The torus amplitude contains a trace over orbifold invariant states. This is ensured by placing a projection operator onto orbifold invariant states inside the trace. The amplitude splits into several terms containing an insertion of an element in the orbifold group. Modular transformation mixes twist sectors and insertions. So, the contribution of twisted sectors is essential. If there are subsets within all contributions to

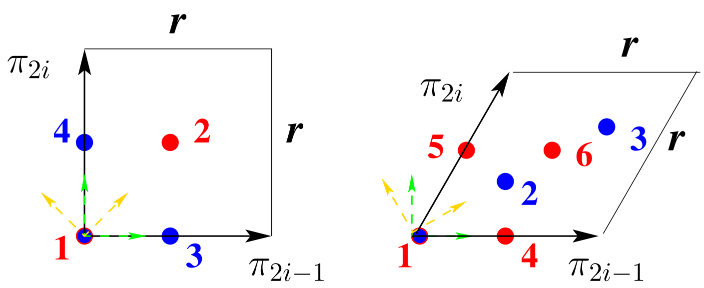

Figure 2: The $\mathbb{Z}_4$ (left) and $\mathbb{Z}_3$ (right) invariant lattices. The $\mathbb{Z}_4$ invariant lattice is the root lattice of $B_2 = SO(5)$ and has two fixed points (1,2) under $\mathbb{Z}_4$ and two further points that are fixed under the $\mathbb{Z}_2$ subsymmetry, but interchanged by the $\mathbb{Z}_4$ action. The $\mathbb{Z}_3$ (and $\mathbb{Z}_6$) invariant lattice is the root lattice of $A_2 = SU(3)$ or equivalently G_2 (observe that the lattices coincide even though the simple roots differ). It has fixed points under the $\mathbb{Z}_3$ subsymmetry (1,2,3) and under the $\mathbb{Z}_2$ subsymmetry (1,4,5,6), where $2 \overset{\mathbb{Z}_6}{\leftrightarrow} 3$ and $4 \overset{\mathbb{Z}_6}{\rightarrow} 5 \overset{\mathbb{Z}_6}{\rightarrow} 6$. The **A**-type lattices have the coordinate orientation x_{2i-1} along the 1-cycle π_{2i-1} (depicted in green); for the **B**-type lattices x_{2i-1} is along $\pi_{2i-1} + \pi_{2i}$ (depicted in yellow).

the torus amplitude which are mapped onto themselves by modular transformations, there can be an arbitrary number in front of the contribution of these subsets. This corresponds to discrete torsion. A simple argument restricting the possible numbers can be found in [80]. Denote the factor not fixed by imposing modular invariance by

$$\epsilon\left(x, y\right), \tag{3}$$

where x and y are in the orbifold group. The expression (3) appears in front of a contribution from the x–twisted sector trace with a y insertion. In the direct computation of the torus amplitude as a trace such ambiguities can be viewed as an unfixed y-eigenvalue of the x-twisted vacuum. As such it should form a representation of the orbifold group, i.e. (z is also an orbifold group element)

$$\epsilon\left(x, y\right) \epsilon\left(x, z\right) = \epsilon\left(x, yz\right). \tag{4}$$

Such a relation leaves a discrete set of possibilities. Let us illustrate that at the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example. Any element in the orbifold group leaves the untwisted vacuum invariant and hence $\epsilon\left(1, \cdots\right) = 1$. Since modular transformations exchange twist and insertion also $\epsilon\left(\cdots, 1\right) = 1$. So, for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold we have

$$\epsilon\left(1, \theta\right) = \epsilon\left(1, \omega\right) = \epsilon\left(\theta, 1\right) = \epsilon\left(\omega, 1\right) = 1. \tag{5}$$

However, the θ eigenvalue of the ω twisted vacuum is not fixed by modular invariance but just related to the ω eigenvalue of the θ twisted vacuum, i.e.

$$\epsilon\left(\theta, \omega\right) = \epsilon\left(\omega, \theta\right) = \eta \tag{6}$$

with a so far undetermined η. Equations (4) and (5) lead finally to

$$\eta = \pm 1 \tag{7}$$

corresponding to the orbifold without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. The situation is tabulated in table 1.

Torus orbits of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$				
twist sector / insertion $\theta^k\omega^l$	00	01	10	11
00	$\times$	$\times$	$\times$	$\times$
01	$\times$	$\times$	$\bullet$	$\bullet$
10	$\times$	$\bullet$	$\times$	$\bullet$
11	$\times$	$\bullet$	$\bullet$	$\times$

Table 1: The two orbits of the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. The twisted sectors (rows) and insertions (columns) are labeled by the powers of the two $\mathbb{Z}_2$ orbifold generators, i.e. kl corresponds to $\theta^k\omega^l$. $\times$ labels the untwisted orbit on which the discrete torsion does not act. $\bullet$ denotes the orbit with discrete torsion $\eta = \pm 1$.

Now, for the $\mathbb{Z}_2 \times \mathbb{Z}_4$ case there are three orbits under modular invariance depicted by crosses, dots and circles in table 2. The orbit with the crosses contains the untwisted sector and hence the corresponding epsilons are one. The orbit with the dot contains $\mathbb{Z}_2$ generators acting on a twisted state. By the same argument as for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case the corresponding epsilons are either all plus or all minus one, corresponding to no discrete torsion or discrete torsion, respectively. To fix the orbit depicted by empty circles we can again use (4), e.g.

$$\epsilon\left(\theta, \omega^2\right) = \epsilon\left(\theta, \omega\right)^2 = \left(\pm 1\right)^2 = 1. \tag{8}$$

So, again there are only two discrete choices.

Finally, the situation for $\mathbb{Z}_2 \times \mathbb{Z}_6$ (and $\mathbb{Z}_2 \times \mathbb{Z}_6'$) is summarised in table 3 . Again there are three orbits of modular transformations denoted by crosses, dots and circles. In the orbit denoted by a cross epsilon is again 1, and for the dotted one we obtain as before $\epsilon = \pm 1$ by, for instance, imposing

$$\epsilon\left(\omega, \theta\right)^2 = \epsilon\left(\omega, 1\right) = 1. \tag{9}$$

The three torus orbits of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$								
twist sector / insertion	00	01	02	03	10	11	12	13
00	×	×	×	×	×	×	×	×
01	×	×	×	×	•	•	•	•
02	×	×	×	×	◯	×	◯	×
03	×	×	×	×	•	•	•	•
10	×	•	◯	•	×	•	◯	•
11	×	•	×	•	•	×	•	×
12	×	•	◯	•	◯	•	×	•
13	×	•	×	•	•	×	•	×

Table 2: The three orbits of the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. The twisted sectors (rows) and insertions (columns) are labeled by the powers of the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold generators, i.e. kl corresponds to $\theta^k \omega^l$. × labels the untwisted orbit on which the torsion does not act. • denotes the orbit with discrete torsion $\eta = \pm 1$. The orbit ◯ transforms as $\eta^2 = 1$.

The epsilon in the orbit denoted by circles is the same as in the one with dots as can be seen from, e.g.

$$\epsilon\left(\theta, \omega^3\right) = \epsilon\left(\theta, \omega\right)^3. \tag{10}$$

The assignment of discrete torsion on the various twist sectors fixes the Hodge numbers of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds on a given lattice completely. Those for factorisable lattices are listed in table 4.

The computation of the $\mathcal{N} = 2$ supersymmetric type IIA closed string spectrum on these orbifolds is discussed in appendix A. The orientifolded $\mathcal{N} = 1$ supersymmetric closed string spectrum, in particular the decomposition of the h_{11} $\mathcal{N} = 2$ vector multiplets into $\mathcal{N} = 1$ chiral (Kähler moduli containing) and vector multiplets, (h_{11}^-, h_{11}^+), depends on the choice of an exotic O6-plane as discussed below. The untwisted and twisted orientifolded closed string spectrum on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ is tabulated in table 44 and table 45, respectively, in appendix A.

The three torus orbits of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$												
twist sector / insertion	00	01	02	03	04	05	10	11	12	13	14	15
00	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
01	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
02	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
03	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bigcirc$	$\bullet$	$\bullet$	$\bigcirc$	$\bullet$	$\bullet$
04	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
05	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
10	$\times$	$\bullet$	$\times$	$\bigcirc$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bigcirc$	$\times$	$\bullet$
11	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$
12	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$
13	$\times$	$\bullet$	$\times$	$\bigcirc$	$\times$	$\bullet$	$\bigcirc$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$
14	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$
15	$\times$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\bullet$	$\times$	$\bullet$	$\times$	$\bullet$	$\times$

Table 3: The three orbits of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. The twisted sectors (rows) and insertions (columns) are labeled by the powers of the $\mathbb{Z}_2$ and $\mathbb{Z}_6$ (or $\mathbb{Z}_6'$) orbifold generators, i.e. kl corresponds to $\theta^k \omega^l$. $\times$ labels the untwisted orbit on which the torsion does not act. $\bullet$ denotes the orbit with discrete torsion $\eta = \pm 1$. The orbit $\bigcirc$ transforms as $\eta^3 = \eta$.

The three-cycles will be discussed in detail in section 3 for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, section 4 for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$, and sections 5 and 6 for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, respectively.

The twist sectors of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ are inherited from various underlying $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ cases as discussed in appendix D. This is on the one hand of interest in view of the fact that intersecting D6-branes on $T^6/\mathbb{Z}_6$ and $T^6/\mathbb{Z}_6'$ have been discussed before in [70, 71] and [68, 69], respectively, and on the other hand when one considers the blow-up of orbifold singularities as discussed e.g. in [81].

So far, we did not discuss orientifolds. There arises another ambiguity, *viz.* eigenvalues of $\Omega\mathcal{R}$. In the following we briefly review what such eigenvalues mean and how they relate

Hodge numbers per twist sector on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ without and with discrete torsion											
$T^6/$ torsion	lattice Hodge numbers	Untwisted	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	$\vec{v}$	$(\vec{v}+\vec{w})$	$(\vec{v}+2\vec{w})$	$(\vec{v}+3\vec{w})$	total
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SU(2)^6$		$(0,\frac{1}{2},-\frac{1}{2})$			$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},0,-\frac{1}{2})$			
$\eta=1$	h_{11}	3	16			16	16			51
	h_{21}	3	0			0	0			3
$\eta=-1$	h_{11}	3	0			0	0			3
	h_{21}	3	16			16	16			51
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$SU(2)^2 \times SO(5)^2$		$(0,\frac{1}{4},-\frac{1}{4})$	$(0,\frac{1}{2},-\frac{1}{2})$		$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{4},-\frac{1}{4})$	$(\frac{1}{2},0,-\frac{1}{2})$		
$\eta=1$	h_{11}	3	8	10		12	16	12		61
	h_{21}	1	0	0		0	0	0		1
$\eta=-1$	h_{11}	3	0	10		4	0	4		21
	h_{21}	1	8	0		0	0	0		1+8
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$SU(2)^2 \times SU(3)^2$		$(0,\frac{1}{6},-\frac{1}{6})$	$(0,\frac{1}{3},-\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{3},-\frac{1}{6})$	$(\frac{1}{2},-\frac{1}{6},-\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
$\eta=1$	h_{11}	3	2	8	6	8	8	8	8	51
	h_{21}	1	0	2	0	0	0	0	0	$1+2$
$\eta=-1$	h_{11}	3	0	8	0	0	4	4	0	19
	h_{21}	1	2	2	6	4	0	0	4	$15+4$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	$SU(3)^3$		$(-\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{6},-\frac{1}{3},\frac{1}{6})$	$(-\frac{1}{6},-\frac{1}{6},\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
$\eta=1$	h_{11}	3	2	9	6	6	2	2	6	36
	h_{21}	0	0	0	0	0	0	0	0	0
$\eta=-1$	h_{11}	3	1	9	0	0	1	1	0	15
	h_{21}	0	0	0	5	5	0	0	5	15

Table 4: Hodge numbers per twist-sector for orbifolds without and with discrete torsion. The three-cycles, which can be wrapped by intersecting D6-branes, stem from the untwisted and various $\mathbb{Z}_2$ twisted sectors and are highlighted in blue. As for the $T^6/\mathbb{Z}_N$ orbifold limits, the total number of three cycles is $b_3 = 2h_{21} + 2$ with the two additional three cycles ($h_{30} = h_{03} = 1$) arising in the untwisted sector.

to discrete torsion. Our arguments follow [82, 33]. An $\Omega\mathcal{R}$ eigenvalue appears in the loop channel and can be ± 1 since $(\Omega\mathcal{R})^2 = 1$. To understand what that means it is useful to relate the loop amplitude to the tree amplitude drawn in figure 3. Figure 4 reviews how this relates to the loop amplitude. To be specific, consider first a particular tree channel amplitude

$$\langle \Omega\mathcal{R}| \, e^{-\ell H_{cl}} \, |\Omega\mathcal{R}\theta^n\omega^m\rangle = \mathrm{Tr}_{\theta^n\omega^m}\left(\Omega\mathcal{R} e^{-2\pi t H}\right), \tag{11}$$

where on the rhs the loop channel form is displayed. First, note that to the tree channel

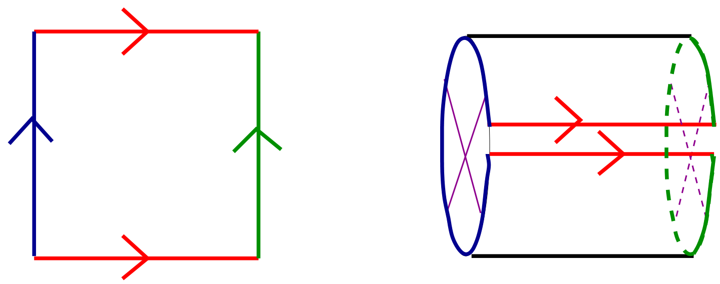

Figure 3: The figure on the left hand side shows the 'triangulated' version of a tree channel Klein Bottle amplitude. In difference to a torus the left and right edges are not glued together but to themselves to form crosscaps as indicated in the figure on the right hand side by purple lines.

amplitude there is no twisted sector closed string contribution since [83]

$$(\Omega\mathcal{R})^2 = (\Omega\mathcal{R}\theta^n\omega^m)^2 = 1. \tag{12}$$

The twist sector of the loop channel can be inferred by closing the path containing a red and brown line in the central drawing in figure 4,

$$(\Omega\mathcal{R})^{-1}\,\Omega\mathcal{R}\theta^n\omega^m = \theta^n\omega^m .$$

The insertion is given by the crosscap map of the left (blue in figure 4) O6-plane, *viz.* $\Omega\mathcal{R}$. The eigenvalue of $\Omega\mathcal{R}$ can be ± 1. Changing it changes signs on both sides of (11). Hence, changing the eigenvalue of $\Omega\mathcal{R}$ in the loop channel corresponds to changing one of the O6-planes in the tree channel to an exotic O6-plane (of opposite charge).

To see how this relates to discrete torsion let $\theta^b\omega^a$ be an orbifold group element acting on the $\theta^n\omega^m$ twisted sector with an additional sign upon turning on discrete torsion ($\eta = -1$). Consider

$$\left\langle \Omega\mathcal{R}\theta^b\omega^a \right| e^{-\ell H_{cl}} \left| \Omega\mathcal{R}\theta^{b+n}\omega^{a+m} \right\rangle = \eta \, \mathrm{Tr}_{\theta^n\omega^m}\left(\Omega\mathcal{R}\theta^b\omega^a e^{-2\pi t H}\right), \tag{13}$$

where the identification of twist sector and insertion works in the same way as before. The effect of discrete torsion has been extracted into η on the rhs of (13). So, turning on discrete torsion changes sign of one of the O6-plane charges on the lhs of (13). Denoting the extra sign of an O6-plane charge by eta with its crosscap map as a subscript we can summarise the situation as [33]

$$\eta_{\Omega\mathcal{R}}\,\eta_{\Omega\mathcal{R}\theta^n\omega^m}\,\eta_{\Omega\mathcal{R}\theta^b\omega^a}\,\eta_{\Omega\mathcal{R}\theta^{b+n}\omega^{a+m}} = \eta, \tag{14}$$

where η is the discrete torsion sign appearing in the torus amplitude element with $\theta^n\omega^m$ twist and $\theta^b\omega^a$ insertion. These considerations generalise the discussion from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case considered in [84, 33] to any $\mathbb{Z}_2 \times \mathbb{Z}_{2M}$. For all orbifolds considered in this article, (14) boils down to the fact that, in the presence of discrete torsion, one (or three) of the four O6-plane orbits under the orbifold action has (have) to be exotic.

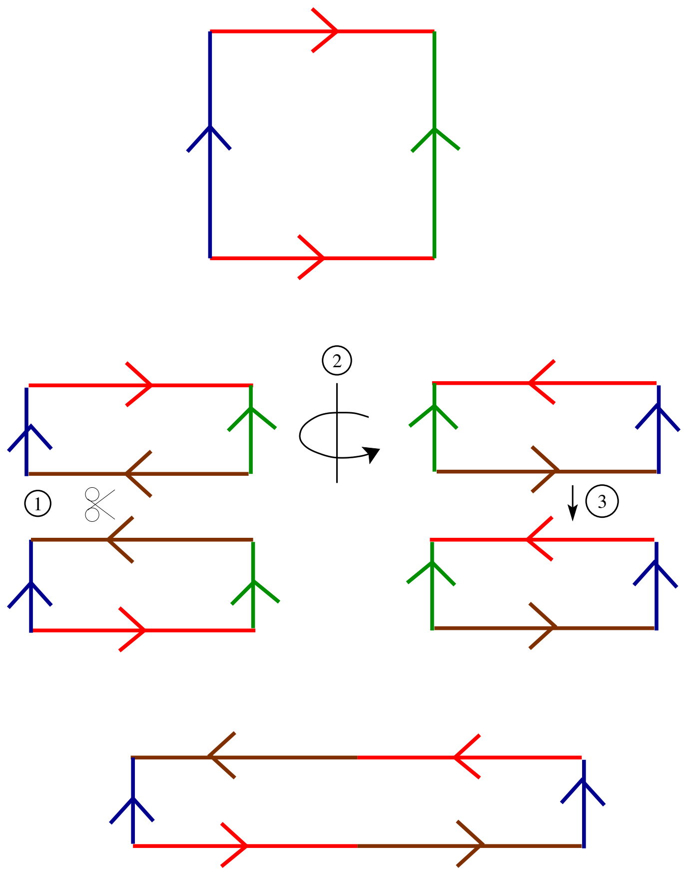

Figure 4: The figure on the top shows the tree channel Klein Bottle amplitude (see figure 3). Topology preserving operations mapping this to the loop channel amplitude are illustrated in the middle: ① cut along the brown line with indicated orientation, ② flip upper rectangle over its right edge, ③ push flipped rectangle down and glue along the green line. Finally, the loop channel diagram is displayed at the bottom. The red and brown line merge into a string closed upon a twist (visible from its tree channel origin). The crosscap map of the blue O6-plane appears as a trace insertion.

2.2 Three-Cycles

Three-cycles within the compact space will be wrapped by D6-branes and O6-planes. Here, we discuss the cycles of interest more explicitly. First, we consider three-cycles on the underlying T^6. These can be written as a direct product of one-cycles π_i (see figures 1 and 2)

$$\Pi^{\rm torus} = \bigotimes_{j=1}^{3} \left(n^j \pi_{2j-1} + m^j \pi_{2j}\right) , \tag{15}$$

where n^j, m^j are co-prime integers denoting wrapping numbers. A bulk cycle on the orbifold is obtained by adding all its orbifold images to the torus cycle. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ maps torus cycles passing through the origin to themselves. Therefore, for such cycles, adding the corresponding images yields an overall factor of four times a sum over images under coset $\mathbb{Z}_2 \times \mathbb{Z}_{2M}/\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)$ representatives. Explicitly one obtains for bulk cycles

$$\Pi^{\text{bulk}} = 4\sum_{m=0}^{M-1} \omega^m \left[\bigotimes_{j=1}^{3} \left(n^j \pi_{2j-1} + m^j \pi_{2j}\right)\right]. \tag{16}$$

Let us discuss first the case of vanishing torsion. From table 4 (and caption) one can see that in this case the third Betti number, b_3, receives contributions only from untwisted sectors (except for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$, which also has a contribution from the $\mathbb{Z}_3$ twisted sector). That means that the dimensionality of the lattice (16) equals b_3. However, on the orbifold the cycles Π^{bulk} form only a sublattice of the integral homology lattice $H_3\left(T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}, \mathbb{Z}\right)$. This can be seen by picking a basis of these bulk cycles and computing the determinant of the intersection form to differ from ± 1. A general lattice vector in $H_3\left(T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}, \mathbb{Z}\right)$ without discrete torsion, i.e. $\eta = 1$, is instead

$$\Pi^{\text{frac}} = \frac{1}{2}\Pi^{\text{bulk}} \tag{17}$$

with integer wrapping numbers n^j, m^j [22]. D6-branes wrapping such a fractional cycle are not completely rigid. Only part of the adjoint moduli is stabilised due to a superpotential [33].

A stack of N D6-branes wrapping four times a torus cycle supports the gauge group $U(4N)$. If the torus cycle passes through the origin, then the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action breaks this gauge group: one $\mathbb{Z}_2$ to $U(2N) \times U(2N)$ which is broken by the second $\mathbb{Z}_2$ to $U(2N)$ [85]. Representatives in the coset $\mathbb{Z}_2 \times \mathbb{Z}_{2M}/\left(\mathbb{Z}_2 \times \mathbb{Z}_2\right)$ fix the gauge group on the corresponding orbifold image stacks. Hence, N D6-branes wrapping a bulk cycle Π^{bulk} support the gauge group $U(2N)$, N D-branes wrapping a fractional cycle (17) carry gauge group $U(N)$.

For the case with discrete torsion there are also twisted contributions to b_3 (see table 4). These additional contributions come from exceptional three-cycles. Given an element of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, a corresponding exceptional cycle wraps an S^1 inherited from the T^2 left invariant under that element and is localised at a fixed point within the other two complex directions. Such a fixed point becomes an S^2 in the blown up version of the orbifold. Explicitly the relevant exceptional three-cycles are

$$\Pi^{\mathbb{Z}_2^{(i)}} = 2\left(-1\right)^{\tau_0^{(i)}} \sum_{m=0}^{M-1} \omega^m \left[\sum_{(\alpha,\beta)\in T_j \times T_k} \left(-1\right)^{\tau_j^{\alpha}+\tau_k^{\beta}} e_{\alpha\beta}^{(i)} \bigotimes \left(n^i \pi_{2i-1} + m^i \pi_{2i}\right)\right], \tag{18}$$

where i labels a T^2 on which $\mathbb{Z}_2^{(i)}$ acts trivially and (i,j,k) are cyclic permutations of $(1,2,3)$. The factor two arises since we take the cycle to pass through the origin of the invariant T^2. Hence, the other $\mathbb{Z}_2 = (\mathbb{Z}_2 \times \mathbb{Z}_2)/\mathbb{Z}_2^{(i)}$ maps the exceptional cycle to itself. The phase factors τ_j^α geometrically correspond to the orientation of the collapsed S^2 and physically are associated to discrete Wilson lines, whereas $(-1)^{\tau_0^{(i)}}$ are the $\mathbb{Z}_2^{(i)}$ eigenvalues with $(-1)^{\tau_0^{(i)}+\tau_0^{(j)}} = (-1)^{\tau_0^{(k)}}$. Including such cycles together with the bulk cycles (16), the dimensionality of the resulting lattice matches b_3 at least for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$. We will briefly comment on the $\mathbb{Z}_4$ exceptional three-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ and the $\mathbb{Z}_6$ and $\mathbb{Z}_3$ exceptional three-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ in section 4 and 5, respectively, none of which can be used for our D6-brane model building purposes.

Following [33], we take a fractional cycle as a superposition of a fractional bulk cycle and fractional exceptional cycles at $\mathbb{Z}_2$ fixed points,

$$\Pi^{\text{frac}} = \frac{1}{4}\left(\Pi^{\text{bulk}} + \sum_{i=1}^{3} \Pi^{\mathbb{Z}_2^{(i)}}\right), \tag{19}$$

where only exceptional cycles through which the bulk cycle passes contribute.

The cycles for $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant fractional D6-branes can be summarised as

$$\Pi^{\text{frac}} = \frac{3+\eta}{8}\,\Pi^{\text{bulk}} + \frac{1-\eta}{8}\sum_{i=1}^{3} \Pi^{\mathbb{Z}_2^{(i)}}, \tag{20}$$

where, as previously, $\eta = \pm 1$ labels the case without and with discrete torsion. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, cycles of the form (20) generate an unimodular lattice; for more details see section 3 and 6, respectively. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$, the cycles (20) generate a sub-lattice of the unimodular lattice of three-cycles, which also contains contributions from $\mathbb{Z}_3^{(1)}$ twisted sectors as discussed in section 5. Finally, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ does not have any three-cycles of the form (20) for $\eta = -1$ since all $\mathbb{Z}_2$ exceptional three-cycles are projected out for any choice of discrete torsion as discussed in section 4, see also table 4. Instead, also for $\eta = -1$, fractional three-cycles are of the form (17) for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$.

To construct orientifold planes one can start with a set of $\Omega\mathcal{R}$ fixed points on T^6 and add orbifold images. Here, however, identical cycles count only once due to the fact that the O6-plane is a non-dynamical object. Schematically we write

$$\Pi_{O6} = \frac{1}{4}\Pi_{O6}^{\text{bulk}}. \tag{21}$$

From identity (12) one learns that there are no twisted sector contributions to the Klein-Bottle. That means that O6-planes do not wrap exceptional cycles. There are no contributions from twisted cycles to the Möbius strip either since O-planes do not carry twisted RR

charges. To the annulus diagram bulk cycles as well as $\mathbb{Z}_2$ twisted cycles can contribute. That other twisted cycles cannot contribute can be seen as follows. The twist sector of the tree-level cylinder diagram becomes an insertion in the annulus amplitude. This insertion should leave the boundary conditions of the open string invariant (otherwise the trace vanishes). But that means that it should leave the D6-branes invariant which is possible only for the identity and a $\mathbb{Z}_2$ generator. To summarise, only fractional bulk cycles and $\mathbb{Z}_2$ twisted cycles enter tadpole cancellation conditions.

In the following section, we discuss how to use the fractional three-cycles (20) and O6-planes (21) for model building.

2.3 Model building with D6-branes in IIA orientifolds

Stacks of N_a identical D6$_a$-branes generically support the gauge groups $U(N_a)$. We discuss the following model building rules for generic Calabi-Yau backgrounds (cf. the review [23]) in terms of the three-cycles Π_a and Π_{O6} wrapped by D6$_a$-branes and O6-planes, respectively, and comment on their explicit form on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ backgrounds:

- The **RR tadpole cancellation conditions** read
$$\sum_a N_a \left(\Pi_a + \Pi_{a'}\right) - 4\,\Pi_{O6} = 0. \tag{22}$$
For $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ backgrounds, the RR tadpole cancellation conditions can be split into bulk and twisted ones,
$$\begin{aligned} \frac{1}{2^{(3-\eta)/2}} \sum_a N_a \left(\Pi_a^{\text{bulk}} + \Pi_{a'}^{\text{bulk}}\right) - 4\,\Pi_{O6} = 0, \\ \sum_a N_a \left(\Pi_a^{\mathbb{Z}_2^{(i)}} + \Pi_{a'}^{\mathbb{Z}_2^{(i)}}\right) = 0 \quad \text{for} \quad i \in \{1,2,3\} \text{ and } 2M \in \{2,6,6'\}. \end{aligned} \tag{23}$$
For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$, only bulk cycles exist, and the prefactor $\frac{1}{2^{(3-\eta)/2}}$ has to be replaced by $\frac{1}{2}$ also for the choice of discrete torsion $\eta = -1$.
- The **chiral spectrum** is given by table 5. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds, the intersection numbers are given in terms of the toroidal wrapping numbers, $\mathbb{Z}_2^{(i)}$ eigenvalues, discrete displacements and Wilson lines,
$$\begin{aligned} \left(\frac{3+\eta}{8}\Pi_a^{\text{bulk}}\right) \circ \left(\frac{3+\eta}{8}\Pi_b^{\text{bulk}}\right) &= -\,2^{\eta-1} \sum_{m=0}^{M-1} I_{a(\omega^m b)}, \\ \left(\frac{1}{4}\Pi_a^{\mathbb{Z}_2^{(i)}}\right) \circ \left(\frac{1}{4}\Pi_b^{\mathbb{Z}_2^{(i)}}\right) &= -\,2^{\eta-1} \sum_{m=0}^{M-1} I_{a(\omega^m b)}^{\mathbb{Z}_2^{(i)}}, \end{aligned} \tag{24}$$

Chiral spectrum	
representation	net chirality
$(\mathbf{N}_a, \overline{\mathbf{N}}_b)$	$\Pi_a \circ \Pi_b$
$(\mathbf{N}_a, \mathbf{N}_b)$	$\Pi_a \circ \Pi_{b'}$
$\mathbf{Anti}_a$	$\frac{1}{2}\left(\Pi_a \circ \Pi_{a'} + \Pi_a \circ \Pi_{O6}\right)$
$\mathbf{Sym}_a$	$\frac{1}{2}\left(\Pi_a \circ \Pi_{a'} - \Pi_a \circ \Pi_{O6}\right)$

Table 5: The chiral spectrum on intersecting D6-branes in terms of intersection numbers of the wrapped three-cycles.

with $I_{ab}^{(j)} \equiv n_a^j m_b^j - m_a^j n_b^j$ and $I_{ab} \equiv \prod_{j=1}^3 I_{ab}^{(j)}$ and the $\mathbb{Z}_2^{(i)}$ invariant intersection number weighted with relative discrete Wilson lines and $\mathbb{Z}_2^{(i)}$ eigenvalue,

$$I_{a(\theta^n\omega^m b)}^{\mathbb{Z}_2^{(i)}} = (-1)^{\tau_{0,a}^{(i)}+\tau_{0,b}^{(i)}} \sum_{\substack{(\alpha_a,\beta_a) \\ (\alpha_b,\beta_b)} \in T_j \times T_k} (-1)^{\tau_j^{\alpha_a}+\tau_k^{\beta_a}+\tau_j^{\alpha_b}+\tau_k^{\beta_b}} \delta_{\alpha_a(\theta^n\omega^m\alpha_b)} \delta_{\beta_a(\theta^n\omega^m\beta_b)} I_{a(\theta^n\omega^m b)}^{(i)}. \quad (25)$$

More details on this expression can be found in appendix A.1 of [79] in the context of $T^6/\mathbb{Z}_{2N}$ models. In order to compute the multiplicites of chiral antisymmetric and symmetric representations, one also needs the intersection number with the orientifold planes,

$$\Pi_a^{\text{frac}} \circ \Pi_{O6} = -2^{\frac{\eta-3}{2}} \sum_{n=0}^{1} \sum_{m=0}^{2M-1} \eta_{\Omega\mathcal{R}\theta^n\omega^m}\,, \quad (26)$$

where $\eta_{\Omega\mathcal{R}\theta^n\omega^m} \equiv \eta_{\Omega\mathcal{R}\theta^n\omega^{m+2}}$ and $\tilde{I}_{a,\Omega\mathcal{R}\theta^n\omega^m} \equiv N_{\Omega\mathcal{R}\theta^n\omega^m}\, I_{a,\Omega\mathcal{R}\theta^n\omega^m}$.
For **orbifolds**, one can go further and compute the **non-chiral spectrum** either by means of Chan-Paton labels of open string states or by using the beta function coefficients which arise in the CFT computation of the gauge thresholds. Details for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ without and with discrete torsion are given in appendix B, where again the discussion follows the one for $T^6/\mathbb{Z}_{2N}$ in [79].

- The **K-theory constraint** states that for any probe D6-brane with $Sp(2)$ gauge group, one has to impose [86]

$$\sum_a N_a \Pi_a \circ \Pi_{Sp\text{-probe}} = 0 \text{ mod } 2. \quad (27)$$

This requires the complete classification of $\Omega\mathcal{R}$ invariant three-cycles on which such probe D6-branes can be wrapped. For each $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ with $2M \in$

$\{2,4,6,6'\}$ on factorisable lattices, we perform the full classification in the corresponding section, and we verify that the three-cycles support $Sp(2M)$ gauge factors, not $SO(2M)$, in appendix B.2.

- **Massless U(1) factors** are linear combinations of various Abelian factors, $U(1)_X = \sum_a c_a U(1)_a$, with $U(1)_a \subset U(N_a)$ and constants c_a. These U(1)s are massless if their effective three-cycle is orientifold invariant,

$$\Pi_{X'} \stackrel{!}{=} \Pi_X \qquad \text{with} \qquad \Pi_X = \sum_a N_a c_a \Pi_a. \tag{28}$$

- The **supersymmetry conditions** state that the D6-branes have to wrap special Lagrangian three-cycles with the same calibration as those wrapped by the O6-planes. In terms of the complex number $\mathcal{Z}_a \equiv \int_{\Pi_a} \Omega_3$ with the holomorphic volume form Ω_3, the supersymmetry conditions can be written as

$$\text{Im}(\mathcal{Z}_a) = 0, \qquad \text{Re}(\mathcal{Z}_a) > 0. \tag{29}$$

For $T^6/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$, the bulk supersymmetry condition can be obtained from

$$\mathcal{Z}_a = \prod_{k=1}^{3} \left(e^{-i\pi\tilde{\phi}_k} x_a^k \right) \tag{30}$$

with $\tilde{\phi}_k$ encoding the orientation of a given two dimensional lattice and x_k^a the 1-cycle wrapped by the D6$_a$-brane on the same lattice,

$$\tilde{\phi}_k = \begin{cases} 0 & \mathbb{Z}_2 : \mathbf{a}/\mathbf{b};\ \mathbb{Z}_4, \mathbb{Z}_3, \mathbb{Z}_6 : \mathbf{A} \\ 1/4 & \mathbb{Z}_4 : \mathbf{B} \\ 1/6 & \mathbb{Z}_3, \mathbb{Z}_6 : \mathbf{B} \end{cases},$$

$$x_a^k = \begin{cases} n_a^k + i\frac{R_2^{(k)}}{R_1^{(k)}} \left(m_a^k + b_k n_a^k \right) & \mathbb{Z}_2 \\ n_a^k + i\, m_a^k & \mathbb{Z}_4 \\ n_a^k + e^{\pi i/3} m_a^k & \mathbb{Z}_3, \mathbb{Z}_6 \end{cases}. \tag{31}$$

The supersymmetry conditions can be verified using the angles of a toroidal three-cycle with respect to the $\Omega\mathcal{R}$ plane expressed in terms of the wrapping numbers

(n_a, m_a) per two-torus,

$$\tan(\pi\phi_a) = \begin{cases} \frac{m_a + b n_a}{n_a}\frac{R_2}{R_1} & \mathbb{Z}_2 : \mathbf{a}, \mathbf{b} \\ \frac{m_a}{n_a} & \mathbb{Z}_4 : \mathbf{A} \\ \frac{m_a - n_a}{m_a + n_a} & \mathbb{Z}_4 : \mathbf{B} \\ \sqrt{3}\frac{m_a}{2n_a + m_a} & \mathbb{Z}_3 : \mathbf{A} \\ \frac{1}{\sqrt{3}}\frac{m_a - n_a}{m_a + n_a} & \mathbb{Z}_3 : \mathbf{B} \end{cases} . \tag{32}$$

In terms of the angles in (32), the first condition in (29) reads $\sum_{i=1}^{3} \tan(\pi\phi_a^{(i)}) - \prod_{i=1}^{3} \tan(\pi\phi_a^{(i)}) = 0$.
A fractional three-cycle is supersymmetric if the bulk part is supersymmetric and only those exceptional cycles are wrapped, whose $\mathbb{Z}_2^{(i)}$ fixed points are trasversed by the bulk cycle. For $T^6/\mathbb{Z}_{2N}$, three of four signs from the fixed point contributions are independent and correspond to the choice of two discrete Wilson lines and the $\mathbb{Z}_2$ eigenvalue. Similarly, on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ with discrete torsion (and $2M \neq 4$), five of the eight signs have the physical interpretation as the choice of three discrete Wilson lines and two independent $\mathbb{Z}_2^{(i)}$ eigenvalues.

3 The $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega\mathcal{R})$ orientifolds on tilted tori

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is generated by the two shift vectors

$$\vec{v} = \frac{1}{2}(1, -1, 0), \qquad \vec{w} = \frac{1}{2}(0, 1, -1), \tag{33}$$

which correspond to $\mathbb{Z}_2^{(3)}$ and $\mathbb{Z}_2^{(1)}$, respectively. The third orbifold twist $\mathbb{Z}_2^{(2)}$ is generated by the sum, $\vec{v} + \vec{w} = \frac{1}{2}(1, 0, -1)$.

On the factorisable lattice $SU(2)^6$, the relevant Hodge numbers for the three-cycles are

$$\eta = 1 : \; h_{21} = 3_{\text{bulk}}, \qquad \eta = -1 : \; h_{21} = 3_{\text{bulk}} + 48_{\mathbb{Z}_2}, \tag{34}$$

where in the case with discrete torsion, the exceptional 3-cycles are evenly distributed over the three $\mathbb{Z}_2^{(i)}$ twisted sectors. The complete list of Hodge numbers per untwisted and

twisted sector is given in table 4 for both choices of discrete torsion.

3.1 The bulk three-cycles

A fractional cycle for the case without discrete torsion is (cf. (15))

$$
\begin{aligned}
\frac{\Pi_a^{\rm bulk}}{2} &= 2 \bigotimes_{j=1}^{3} \left(n^j \pi_{2j-1} + m^j \pi_{2j} \right) \\
&= 2 \left(n_a^1 n_a^2 n_a^3 \Pi_{135} + \sum_{i=1}^{3} n_a^i m_a^j m_a^k \Pi_{2i-1;2j;2k} + m_a^1 m_a^2 m_a^3 \Pi_{246} \right. \\
&\qquad \left. + \sum_{i=1}^{3} m_a^i n_a^j n_a^k \Pi_{2i;2j-1;2k-1} \right).
\end{aligned} \tag{35}
$$

Again, (i,j,k) is a cyclic permutation of $(1,2,3)$. The appearing three-cycles are direct products of three one-cycles as encoded in the triple index. The index a labels a stack of fractional D6-branes wrapping the three-cycle. For the case with discrete torsion (half) the above cycle contributes as in (20).

A D6-brane wrapping the cycle (35) has to be accompanied by its orientifold image wrapping the cycle

$$
\begin{aligned}
\frac{1}{2}\Pi_{a'}^{\rm bulk} = 2 &\left(\left(\prod_{i=1}^{3} n_a^i \right) \Pi_{135} + \sum_{i=1}^{3} n_a^i \left(m_a^j + 2b_j n_a^j \right) \left(m_a^k + 2b_k n_a^k \right) \Pi_{2i-1;2j;2k} \right. \\
&\left. - \left(\prod_{i=1}^{3} \left(m_a^i + 2b_i n_a^i \right) \right) \Pi_{246} - \sum_{i=1}^{3} \left(m_a^i + 2b_i n_a^i \right) n_a^j n_a^k \Pi_{2i;2j-1;2k-1} \right).
\end{aligned} \tag{36}
$$

Here, it matters whether an underlying T^2 is of $\mathbf{a}$ or $\mathbf{b}$ orientation (see figure 1). If the $i^{\rm th}$ two-torus is of the $\mathbf{a}$ type then $b_i = 0$, otherwise $b_i = 1/2$.

The supersymmetry conditions (29) can be written as follows

$$
\begin{aligned}
\sum_{k=1}^{3} \frac{1}{\varrho_k} n_a^i n_a^j (m_a^k + b_k n_a^k) - \prod_{i=1}^{3} (m_a^i + b_i n_a^i) = 0, \\
n_a^1 n_a^2 n_a^3 - \sum_{i=1}^{3} \varrho_i \, n_a^i (m_a^j + b_j n_a^j)(m_a^k + b_k n_a^k) > 0,
\end{aligned} \tag{37}
$$

with $\varrho_k \equiv \frac{R_2^{(i)}}{R_1^{(i)}} \frac{R_2^{(j)}}{R_1^{(j)}}$ and (i,j,k) cyclic permutations of $(1,2,3)$ as before.

For the orientifold fixed cycles there is again a difference between **a** and **b** lattices. Consider figure 1. For the left torus horizontal lines passing through the origin as well as through points shifted vertically by half a lattice vector are invariant under complex conjugation, whereas for the right torus only the horizontal line passing through the origin consists of fixed points under complex conjugation. Therefore, it is useful to introduce the numerical factor

$$N_{O6} = 2^3 \prod_{i=1}^{3} (1 - b_i)\,. \tag{38}$$

The cycle fixed under complex conjugations, i.e. $\mathcal{R}$, is

$$\Pi_{\Omega\mathcal{R}} = \left(\prod_{i=1}^{3} \frac{1}{1-b_i} \right) \left(\Pi_{135} - \sum_{i=1}^{3} b_i \Pi_{2i;2j-1;2k-1} + \sum_{k=1}^{3} b_i b_j \Pi_{2i;2j;2k-1} - b_1 b_2 b_3\, \Pi_{246} \right). \tag{39}$$

Next, we specify the set fixed under $\mathcal{R}\mathbb{Z}_2^{(k)}$, where $\mathbb{Z}_2^{(k)}$ is an element in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold group leaving the k^{th} two-torus invariant and (i,j,k) are cyclic permutations of $(1,2,3)$,

$$\Pi_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}} = \frac{1}{1-b_k} \left(-\Pi_{2i;2j;2k-1} + b_k \Pi_{246} \right). \tag{40}$$

With these ingredients, we can write down the overall cycle wrapped by the O6-plane in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold,

$$\begin{aligned}
\Pi_{O6} &\equiv N_{O6} \left(\eta_{\Omega\mathcal{R}} \Pi_{\Omega\mathcal{R}} + \sum_{i=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} \Pi_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} \right) \\
&= 8 \Bigg(\eta_{\Omega\mathcal{R}}\, \Pi_{135} - \eta_{\Omega\mathcal{R}} \sum_{i=1}^{3} b_i\, \Pi_{2i;2j-1;2k-1} \\
&\qquad + \sum_{k=1}^{3} \left\{ b_i b_j \eta_{\Omega\mathcal{R}} - (1-b_i)(1-b_j) \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}} \right\} \Pi_{2i;2j;2k-1} \\
&\qquad + \left\{ -b_1 b_2 b_3\, \eta_{\Omega\mathcal{R}} + \sum_{k=1}^{3} (1-b_i)(1-b_j) b_k\, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}} \right\} \Pi_{246} \Bigg).
\end{aligned} \tag{41}$$

If there is no discrete torsion, a consistent option is to take none of the O6-planes to be exotic. With discrete torsion, one needs an odd number of exotic O6-planes, due to the relation (14) which can be rewritten in this case as

$$\eta_{\Omega\mathcal{R}} \prod_{k=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}} = \eta. \tag{42}$$

The O6-planes are affected by discrete torsion only by these signs. Because of (12) they do not wrap exceptional cycles [32].

The untwisted RR tadpole cancellation conditions (22) on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ are explicitly

$$\begin{aligned} 0 = &\left[\sum_a N_a n_a^1 n_a^2 n_a^3 - 2^{\frac{7-\eta}{2}} \, \eta_{\Omega\mathcal{R}} \right] \\ &\quad \times \left(\Pi_{1,3,5} + \sum_{\{i,j,k\} \text{ cyclic}} \left(b_j b_k \Pi_{2i-1,2j,2k} - b_i \Pi_{2i,2j-1,2k-1} \right) - b_1 b_2 b_3 \Pi_{2,4,6} \right) \\ &+ \sum_{\{i,j,k\} \text{ cyclic}} \left[\sum_a N_a n_a^i \left(m_a^j + b_j n_a^j \right) \left(m_a^k + b_k n_a^k \right) + 2^{\frac{7-\eta}{2}} \, (1-b_j)(1-b_k) \, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} \right] \\ &\quad \times \left(\Pi_{2i-1,2j,2k} - b_i \Pi_{2,4,6} \right) . \end{aligned} \tag{43}$$

3.2 The exceptional three-cycles

The discrete torsion factor η acts by adding a phase to the $\mathbb{Z}_2^{(i)}$ projection on the $\mathbb{Z}_2^{(k)}$ twisted sectors according to table 1. This can be summarised as

$$e_{\alpha\beta}^{(k)} \xrightarrow{\mathbb{Z}_2^{(i)}} \eta \, e_{\alpha\beta}^{(k)}, \qquad \left\{ \begin{array}{r} \pi_{2k-1} \xrightarrow{\mathbb{Z}_2^{(i)}} -\pi_{2k-1} \\ \pi_{2k} \xrightarrow{\mathbb{Z}_2^{(i)}} -\pi_{2k} \end{array} \right. \quad \text{for } i \neq k, \tag{44}$$

where $\alpha\beta$ labels a $\mathbb{Z}_2^{(k)}$ fixed point on $T_{(i)}^2 \times T_{(j)}^2$. The orbifold invariant three-cycles are obtained by tensoring the exceptional two-cycles with toroidal one-cycles,[4]

$$\begin{aligned} \varepsilon_{\alpha\beta}^{(k)} &\equiv 2 \, e_{\alpha\beta}^{(k)} \otimes \pi_{2k-1} \xrightarrow{\mathbb{Z}_2^{(i)}} -\eta \, \varepsilon_{\alpha\beta}^{(k)}, \\ \tilde{\varepsilon}_{\alpha\beta}^{(k)} &\equiv 2 \, e_{\alpha\beta}^{(k)} \otimes \pi_{2k} \xrightarrow{\mathbb{Z}_2^{(i)}} -\eta \, \tilde{\varepsilon}_{\alpha\beta}^{(k)}. \end{aligned} \tag{45}$$

In the absence of discrete torsion, $\eta = 1$, the two-cycles (44) survive the orbifold projections from the other sectors, whereas with discrete torsion, $\eta = -1$, the three-cycles (45) survive. In the following, we will concentrate on this latter case.

The intersection number among exceptional three-cycles reads

$$\varepsilon_{\alpha\beta}^{(k)} \circ \tilde{\varepsilon}_{\alpha'\beta'}^{(k')} = -4 \, \delta^{kk'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \tag{46}$$

and all others vanish.

[4] In [33], these cycles were called $\alpha_{,n}$ and $\alpha_{,m}$. However, throughout this article, we stick to the notation that $\mathbb{Z}_2$ exceptional three-cycles are labeled by ε and $\tilde{\varepsilon}$ with intersection numbers $\varepsilon \circ \varepsilon = \tilde{\varepsilon} \circ \tilde{\varepsilon} = 0$.

The orientifold images of the exceptional three-cycles are derived by combining the action of $\mathcal{R}$ on the $\mathbb{Z}_2^{(i)}$ fixed points and toroidal one-cycles with the signs from Ω and the exotic O6-plane,

$$\eta_{(k)} \equiv \eta_{\Omega\mathcal{R}} \cdot \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}}, \tag{47}$$

on the exceptional cycles as discussed around equation (13). The results are given in table 6.

$\Omega\mathcal{R}$ **projection on exceptional 3-cycles on** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$	
$\varepsilon^{(k)}_{\alpha\beta}$	$\eta_{(k)} \left(-\varepsilon^{(k)}_{\mathcal{R}(\alpha\beta)} + 2\, b_k\, \tilde{\varepsilon}^{(k)}_{\mathcal{R}(\alpha\beta)} \right)$
$\tilde{\varepsilon}^{(k)}_{\alpha\beta}$	$\eta_{(k)}\, \tilde{\varepsilon}^{(k)}_{\mathcal{R}(\alpha\beta)}$

Table 6: The orientifold projection on exceptional three-cycles on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega\mathcal{R})$ with discrete torsion preserves the twist sector. The $\mathbb{Z}_2^{(k)}$ fixed points $(\alpha\beta)$ on $T_i \times T_j$ transform under $\mathcal{R}$ as described in the caption of figure 1. The signs $\eta_{(k)}$ are as defined in (47).

It is often convenient to work with the sign factors $\eta_{(k)}$ of the orientifold projection on $\mathbb{Z}_2^{(k)}$ twisted sectors, which are in one-to-one correspondence with the signs of the O6-planes, see table 7.

Relation between the sets of signs	
$(\eta_{\Omega\mathcal{R}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}})$	$(\eta_{(1)}, \eta_{(2)}, \eta_{(3)})$
$(-1, 1, 1, 1)$	$(-1. -1. -1)$
$(1, -1, 1, 1)$	$(-1, 1, 1)$
$(1, 1, -1, 1)$	$(1, -1, 1)$
$(1, 1, 1, -1)$	$(1, 1, -1)$

Table 7: Relation between the signs of exotic O6-planes and the orientifold projection on $\mathbb{Z}_2$ twisted sectors for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ with discrete torsion. Since supersymmetric model building only admits one exotic O6-plane, there is a 1-1 correspondence between the sets $(\eta_{\Omega\mathcal{R}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}}, \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}})$ and $(\eta_{(1)}, \eta_{(2)}, \eta_{(3)})$.

On $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with discrete torsion, the exceptional cycles (18) can be written as

$$\Pi^{\mathbb{Z}_2^{(k)}} = \sum_{(\alpha,\beta)\in T_i \times T_j} \left(x^{(k)}_{\alpha\beta}\, \varepsilon^{(k)}_{\alpha\beta} + y^{(k)}_{\alpha\beta}\, \tilde{\varepsilon}^{(k)}_{\alpha\beta} \right), \tag{48}$$

where

$$\begin{aligned}
x^{(k)}_{\alpha_1\beta_1} &= (-1)^{\tau_0^{(k)}}\, n^k, & x^{(k)}_{\alpha_2\beta_1} &= (-1)^{\tau_0^{(k)}+\tau_i}\, n^k, \\
x^{(k)}_{\alpha_1\beta_2} &= (-1)^{\tau_0^{(k)}+\tau_j}\, n^k, & x^{(k)}_{\alpha_2\beta_2} &= (-1)^{\tau_0^{(k)}+\tau_i+\tau_j}\, n^k, \\
y^{(k)}_{\alpha_1\beta_1} &= (-1)^{\tau_0^{(k)}}\, m^k, & y^{(k)}_{\alpha_2\beta_1} &= (-1)^{\tau_0^{(k)}+\tau_i}\, m^k, \\
y^{(k)}_{\alpha_1\beta_2} &= (-1)^{\tau_0^{(k)}+\tau_j}\, m^k, & y^{(k)}_{\alpha_2\beta_2} &= (-1)^{\tau_0^{(k)}+\tau_i+\tau_j}\, m^k,
\end{aligned} \tag{49}$$

and $\tau_0^{(k)} \in \{0,1\}$ parameterises the $\mathbb{Z}_2^{(k)}$ eigenvalue, whereas $\tau_i, \tau_j \in \{0,1\}$ correspond to the choice of discrete Wilson lines on $T^2_{(i)} \times T^2_{(j)}$.

The twisted RR tadpole cancellation conditions take the form

$$0 = \sum_{k=1}^{3} \sum_{(\alpha,\beta)\in T_i\times T_j} \left\{ x^{(k)}_{\alpha\beta} \left(\varepsilon^{(k)}_{\alpha\beta} - \eta_{(k)}\, \varepsilon^{(k)}_{\mathcal{R}(\alpha\beta)} + 2\, b_k\, \eta_{(k)}\, \tilde{\varepsilon}^{(k)}_{\mathcal{R}(\alpha\beta)} \right) + y^{(k)}_{\alpha\beta} \left(\tilde{\varepsilon}^{(k)}_{\alpha\beta} + \eta_{(k)}\, \tilde{\varepsilon}^{(k)}_{\mathcal{R}(\alpha\beta)} \right) \right\},$$

where $\mathcal{R}(\alpha\beta)$ is the orientifold image of the fixed point $\alpha\beta$.

3.3 The K-theory constraint

For discrete torsion and the **aaa** torus, the K-theory constraint has been derived before in [33]. We will generalise their result to an arbitrary choice of tilted tori.

As a first step, we classify all cycles which are topologically $\Omega\mathcal{R}$ invariant and whose bulk part is parallel to some $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ invariant plane, where we shorten the notation by setting $\Omega\mathcal{R} \equiv \Omega\mathcal{R}\mathbb{Z}_2^{(0)}$. The bulk part of these cycles is the same as one of the O6-planes in (39) and (40). In table 8, we compute the bulk contributions to the K-theory constraints explicitly, where the last column shows the simplified result after the untwisted RR tadpole cancellation conditions (43) have been inserted. One can read off that e.g. for the case without discrete torsion and some tilted torus $b_k = 1/2$, the two K-theory constraints associated to the probe branes $Sp(2)_0$ and $Sp(2)_k$ become trivial. Note, that in the case without discrete torsion ($\eta = 1$) at least one of the two-tori should be tilted for interesting models with three generations [53, 54].

Bulk parts of K-theory constraints on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$			
			after RR tcc - mod 2
$\Omega\mathcal{R}$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_0}^{\text{bulk}}}{2^{3-\eta}}$	$\frac{\sum_a N_a \prod_{i=1}^3 (m_a^i + b_i n_a^i)}{2^{1-\eta} \prod_{i=1}^3 (1-b_i)}$	$\frac{\sum_a N_a \prod_{i=1}^3 m_a^i - \sum_a N_a \, b_j b_k \, m_a^i n_a^j n_a^k}{2^{1-\eta} \prod_{i=1}^3 (1-b_i)}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(k)}$ $k=1,2,3$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_k}^{\text{bulk}}}{2^{3-\eta}}$	$\frac{-\sum_a N_a n_a^i n_a^j (m_a^k + b_k n_a^k)}{2^{1-\eta} \, (1-b_k)}$	$\frac{-\sum_a N_a n_a^i n_a^j m_a^k}{2^{1-\eta} \, (1-b_k)}$

Table 8: Bulk contribution to the K-theory constraint on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. In the last column, the result has been simplified by applying the untwisted RR tadpole cancellation conditions (43).

The invariant exceptional three-cycles have the toroidal one-cycle either parallel to the $\Omega\mathcal{R}$ invariant plane (*'horizontal'*),

$$\begin{aligned}\Pi_h^{\mathbb{Z}_2^{(k)}} = \Pi_{h'}^{\mathbb{Z}_2^{(k)}} &= (-1)^{\tau_0^{(k)}} \left(\varepsilon_{\alpha_1\beta_1}^{(k)\,\prime} + (-1)^{\tau_i} \varepsilon_{\alpha_2\beta_1}^{(k)\,\prime} + (-1)^{\tau_j} \varepsilon_{\alpha_1\beta_2}^{(k)\,\prime} + (-1)^{\tau_i+\tau_j} \varepsilon_{\alpha_2\beta_2}^{(k)\,\prime} \right) \\ &\stackrel{!}{=} -\eta_{(k)} (-1)^{\tau_0^{(k)}} \left(\varepsilon_{\mathcal{R}(\alpha_1\beta_1)}^{(k)\,\prime} + (-1)^{\tau_i} \varepsilon_{\mathcal{R}(\alpha_2\beta_1)}^{(k)\,\prime} + (-1)^{\tau_j} \varepsilon_{\mathcal{R}(\alpha_1\beta_2)}^{(k)\,\prime} + (-1)^{\tau_i+\tau_j} \varepsilon_{\mathcal{R}(\alpha_2\beta_2)}^{(k)\,\prime} \right),\end{aligned} \tag{50}$$

where for brevity we defined $\varepsilon_{\alpha\beta}^{(k)\,\prime} \equiv \frac{1}{1-b_k} (\varepsilon_{\alpha\beta}^{(k)} - b_k \, \tilde{\varepsilon}_{\alpha\beta}^{(k)})$, or are at angle $\pi/2$ to the $\Omega\mathcal{R}$ invariant plane (*'vertical'*),

$$\begin{aligned}\Pi_v^{\mathbb{Z}_2^{(k)}} = \Pi_{v'}^{\mathbb{Z}_2^{(k)}} &= (-1)^{\tau_0^{(k)}} \left(\tilde{\varepsilon}_{\alpha_1\beta_1}^{(k)} + (-1)^{\tau_i} \tilde{\varepsilon}_{\alpha_2\beta_1}^{(k)} + (-1)^{\tau_j} \tilde{\varepsilon}_{\alpha_1\beta_2}^{(k)} + (-1)^{\tau_i+\tau_j} \tilde{\varepsilon}_{\alpha_2\beta_2}^{(k)} \right) \\ &\stackrel{!}{=} \eta_{(k)} (-1)^{\tau_0^{(k)}} \left(\tilde{\varepsilon}_{\mathcal{R}(\alpha_1\beta_1)}^{(k)} + (-1)^{\tau_i} \tilde{\varepsilon}_{\mathcal{R}(\alpha_2\beta_1)}^{(k)} + (-1)^{\tau_j} \tilde{\varepsilon}_{\mathcal{R}(\alpha_1\beta_2)}^{(k)} + (-1)^{\tau_i+\tau_j} \tilde{\varepsilon}_{\mathcal{R}(\alpha_2\beta_2)}^{(k)} \right).\end{aligned} \tag{51}$$

In order to classify those fractional three-cycles,

$$\begin{aligned}\Pi_{\text{probe},0} &\equiv \frac{1}{4} \Pi_{\Omega\mathcal{R}}^{\text{bulk}} + \frac{1}{4} \sum_{k=1}^{3} \Pi_h^{\mathbb{Z}_2^{(k)}}, \\ \Pi_{\text{probe},i} &\equiv \frac{1}{4} \Pi_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}^{\text{bulk}} + \frac{1}{4} \left(\Pi_h^{\mathbb{Z}_2^{(i)}} + \Pi_v^{\mathbb{Z}_2^{(j)}} - \Pi_v^{\mathbb{Z}_2^{(k)}} \right),\end{aligned} \tag{52}$$

which are $\Omega\mathcal{R}$ invariant, one needs to evaluate the orientifold images of the orbifold fixed points $\alpha\beta$ at which the exceptional three-cycles are stuck, see table 9.

$\mathbb{Z}_2$ **fixed points for** $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ **invariant branes on** $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$		
(σ)	α_1, β_1	α_2, β_2
$h(0)$	1	$\frac{2}{1-b}$
$h(1)$	$4(1-b)$	3
$v(0)$	1	4
$v(1)$	2	3

Table 9: $\mathbb{Z}_2$ fixed points which are trasversed by a one-cycle parallel to the $\Omega\mathcal{R}$ invariant plane (h) or perpendicular to it (v). The argument σ parameterises the discrete displacement of the one-cycle on the same two-torus.

The $\mathbb{Z}_2$ fixed points per two-torus transform under the orientifold projection as follows,

$$\begin{aligned} 1 &\stackrel{\Omega\mathcal{R}}{\longrightarrow} 1, \\ \frac{2}{1-b} &\stackrel{\Omega\mathcal{R}}{\longrightarrow} \frac{2}{1-b}, \\ 3 &\stackrel{\Omega\mathcal{R}}{\longrightarrow} 3-2b, \\ 4(1-b) &\stackrel{\Omega\mathcal{R}}{\longrightarrow} 4-2b. \end{aligned} \tag{53}$$

A careful analysis of all combinations of untilted and tilted tori with discrete displacements from the origin in table 50 leads to the complete list of invariance conditions in table 51 in the appendix, which can in short be summarised as the relations among displacements σ_i, Wilson lines τ_i, the shape of the tori b_i and the orientifold projection $\eta_{(k)}$ on twisted sectors for $i \neq k$ in table 10.

The $\Omega\mathcal{R}$ invariant branes can either carry $SO(2M)$ or $Sp(2M)$ gauge factors. The analysis in appendix B.2 reveals that the complete classification of $\Omega\mathcal{R}$ invariant branes provides $Sp(2M)$ gauge factors.

3.4 A $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ example with discrete torsion on a tilted torus

Here, we present an example with discrete torsion inspired by [33]. We take the horizontal O-plane to be exotic and all the other ones non-exotic. In difference to [33] we choose the **aab** lattice for compactification, i.e. we tilt the third two-torus. In our brane setup we aim

Existence of $\Omega\mathcal{R}$ invariant three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$	
parallel to O6-plane	$(\eta_{(1)}, \eta_{(2)}, \eta_{(3)}) \stackrel{!}{=}$
$\Omega\mathcal{R}$	$\big(-(-1)^{2(b_2\sigma_2\tau_2+b_3\sigma_3\tau_3)}, -(-1)^{2(b_1\sigma_1\tau_1+b_3\sigma_3\tau_3)}, -(-1)^{2(b_1\sigma_1\tau_1+b_2\sigma_2\tau_2)}\big)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$\big(-(-1)^{2(b_2\sigma_2\tau_2+b_3\sigma_3\tau_3)}, (-1)^{2(b_1\sigma_1\tau_1+b_3\sigma_3\tau_3)}, (-1)^{2(b_1\sigma_1\tau_1+b_2\sigma_2\tau_2)}\big)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$\big((-1)^{2(b_2\sigma_2\tau_2+b_3\sigma_3\tau_3)}, -(-1)^{2(b_1\sigma_1\tau_1+b_3\sigma_3\tau_3)}, (-1)^{2(b_1\sigma_1\tau_1+b_2\sigma_2\tau_2)}\big)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$\big((-1)^{2(b_2\sigma_2\tau_2+b_3\sigma_3\tau_3)}, (-1)^{2(b_1\sigma_1\tau_1+b_3\sigma_3\tau_3)}, -(-1)^{2(b_1\sigma_1\tau_1+b_2\sigma_2\tau_2)}\big)$

Table 10: Conditions on the existence of $\Omega\mathcal{R}$ invariant fractional three-cycles on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ with discrete torsion for $2M \in \{2, 6, 6'\}$ using the relations in table 51.

Bulk cycles for a $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ example with discrete torsion				
N_α	(n^1_α, m^1_α)	(n^2_α, m^2_α)	$(n^3_\alpha, \tilde{m}^3_\alpha)$	$m^3_\alpha = \tilde{m}^3_\alpha - \frac{1}{2}n^3_\alpha$
$N_{a_1} = 2$	$(-1, 1)$	$(-2, 1)$	$(-1, \frac{1}{2})$	1
$N_{a_2} = 2$	$(1, -1)$	$(2, -1)$	$(-1, \frac{1}{2})$	1
$N_{a_3} = 2$	$(-1, 1)$	$(2, -1)$	$(1, -\frac{1}{2})$	-1
$N_{a_4} = 2$	$(1, -1)$	$(-2, 1)$	$(1, -\frac{1}{2})$	-1
$N_{b_1} = 2$	$(1, 0)$	$(0, 1)$	$(0, -1)$	-1
$N_{b_2} = 2$	$(-1, 0)$	$(0, 1)$	$(0, 1)$	1
$N_{c_1} = 2$	$(0, 1)$	$(0, -1)$	$(2, 0)$	-1
$N_{c_2} = 2$	$(0, 1)$	$(0, 1)$	$(-2, 0)$	1

Table 11: Solution to tadpole conditions for tilted $T^2_{(3)}$ (i.e. $b_3 = \frac{1}{2}$), exceptional $\Omega\mathcal{R}$ plane, the $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ planes are regular.

for simplicity and not to get a phenomenologically relevant model. The brane configuration to be discussed is given in table 11.

All branes pass through the origin and have trivial discrete Wilson lines. To specify the $\tau_0^{(i)}$'s we adopted the compressed notation of [33]. For instance, all branes labeled by

a_i, $i = 1, \ldots, 4$ wrap the same fractional bulk cycle but differ in their $\tau_0^{(i)}$'s. The $\tilde{m}^i$'s are useful numbers for computing tadpole cancellation and torus intersection numbers whereas the numbers denoted by m^i should be integers. The latter ones should be used when determining through which fixed points the corresponding brane passes. Note, for instance, that for the $\Omega\mathcal{R}$ image of the branes labeled by a_i, $i = 1, \ldots, 4$, the wrapping number m^3 vanishes. This reflects the earlier observation that on a tilted torus the second and third fixed points are interchanged by $\Omega\mathcal{R}$. Naively, the gauge group is $U(4)^4 \times U(2)^2 \times U(2)^2$. Anomalous $U(1)$'s become massive via the Green–Schwarz mechanism leaving as a non-anomalous gauge group $SU(2)^4 \times SU(2)^2 \times SU(2)^2$, as we will see shortly. Supersymmetry conditions on the torus moduli arise from the a-type branes and are

$$\varrho_1 = 4 + \frac{2\varrho_1}{\varrho_3 + \varrho_2}, \tag{54}$$

from $\mathrm{Im}(\mathcal{Z}) = 0$, and then $\mathrm{Re}(\mathcal{Z}) > 0$ for any positive radii.

The model gives rise to a spectrum of chiral multiplets transforming under $U(2)^8$ as follows (the ordering of the group factors follows the ordering of the branes in table 11)

$$\begin{aligned}
&2\left(\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right) + 2\left(\mathbf{1},\overline{\mathbf{2}},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right) + 2\left(\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1}\right)\\
&+2\left(\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}\right) + 2\left(\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}\right) + 2\left(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1}\right)\\
&+2\left(\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}}\right) + 2\left(\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}\right) + 2\left(\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1}\right)\\
&+2\left(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}\right) + 2\left(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\mathbf{2},\mathbf{1}\right) + 2\left(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}\right)\\
&+2\left(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\overline{\mathbf{2}},\mathbf{1}\right) + 2\left(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{2}},\mathbf{1},\overline{\mathbf{2}}\right) + 6\left(\overline{\mathbf{3}}_S,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right)\\
&+6\left(\mathbf{1},\overline{\mathbf{3}}_S,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right) + 6\left(\mathbf{1},\mathbf{1},\overline{\mathbf{3}}_S,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right) + 6\left(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{3}}_S,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}\right).
\end{aligned}$$

Here, the subscript S denotes a symmetric representation of $U(2)$. So far, we have listed the chiral spectrum. It can be seen that the normal $U(1)$ subgroups within each $U(2)$ are anomalous. The corresponding $U(1)$ gauge fields become massive via the Green–Schwarz mechanism. In addition to the chiral spectrum there can be non-chiral matter as discussed in appendix B: For each a_ia_j, b_ib_j, c_ic_j pair with $i \neq j$ there is an additional non-chiral pair transforming in the bifundamental of the corresponding $U(2)^2$, and for each stack b_i, c_i there is a non-chiral pair of antisymmetric representations of the corresponding $U(2)$ factor.

4 The $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ orientifolds

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold is generated by the two shift vectors

$$\vec{v} = \frac{1}{2}(1, -1, 0), \quad \vec{w} = \frac{1}{4}(0, 1, -1). \tag{55}$$

On the factorisable lattice $SU(2)^2 \times SO(5)^2$, the number of three-cycles is given by

$$\eta = 1: \; h_{21} = 1_{\text{bulk}}, \qquad\qquad \eta = -1: \; h_{21} = 1_{\text{bulk}} + 8_{\mathbb{Z}_4}. \tag{56}$$

The Hodge number h_{11} can be found in table 4.

4.1 Model building on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ without and with discrete torsion

For both cases, without and with torsion, only the four independent bulk three-cycles

$$\rho_1 = 4\left(\pi_{135} - \pi_{146}\right), \; \rho_2 = 4\left(\pi_{236} + \pi_{245}\right), \; \rho_3 = 4\left(\pi_{246} - \pi_{235}\right), \; \rho_4 = 4\left(\pi_{145} + \pi_{136}\right), \tag{57}$$

with non-vanishing intersection numbers

$$\rho_1 \circ \rho_3 = \rho_2 \circ \rho_4 = -4 \tag{58}$$

can be used for model building. This orbifold differs from the other ones discussed in this article in that D6-branes wrap the same half-bulk three-cycles independently of the choice of discrete torsion. The case without discrete torsion was first investigated in [64, 65], and in [67] it was shown that three Standard Model generations cannot be obtained. The number theoretic proof used the minimal requirements of (a) three intersections of the QCD stack with a second stack, (b) no symmetric representations on the QCD stack and (c) supersymmetry. These constraints are common to all Standard Model, Pati-Salam and $SU(5)$ GUT models on any Calabi-Yau or orbifold background. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$, the three-cycles on which D6-branes can be wrapped do not depend on the choice of torsion, and therefore conditions (a) and (c) are the same for both choices of $\eta = \pm 1$. The discrete torsion enters only in condition (b) in terms of the sign of one of the O6-planes. In [66], the no-go theorem for the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold without discrete torsion was confirmed for some examples.

For the sake of completeness of discussing $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds on factorisable tori, we give in the following all model building ingredients.

The $\mathbb{Z}_4$ orbifold permutes wrapping numbers on the second and third torus,

$$\begin{pmatrix} n^1 & m^1 \\ n^2 & m^2 \\ n^3 & m^3 \end{pmatrix} \stackrel{\mathbb{Z}_4^{(1)}}{\rightarrow} \begin{pmatrix} n^1 & m^1 \\ -m^2 & n^2 \\ m^3 & -n^3 \end{pmatrix}, \tag{59}$$

and a generic orbifold invariant three-cycle is parameterised by

$$
\begin{aligned}
\Pi^{\text{bulk}} =& P\,\rho_1 + Q\,\rho_2 + U\,\rho_3 + V\,\rho_4, \\
& P = n^1\,\left(n^2 n^3 - m^2 m^3\right), \\
& Q = m^1\,\left(n^2 m^3 + m^2 n^3\right), \\
& U = m^1\,\left(m^2 m^3 - n^2 n^3\right), \\
& V = n^1\,\left(n^2 m^3 + m^2 n^3\right).
\end{aligned} \tag{60}
$$

The supersymmetry conditions read

$$
\begin{array}{lll}
\textbf{a/bAA}: & V + \varrho(-U + b\,P) = 0, \qquad P - \varrho(Q + b\,V) > 0, \\
\textbf{a/bAB}: & [V - P] + \varrho(-U + Q + b\,[P + V]) = 0, \qquad P + V - \varrho(Q + U + b\,[V - P]) > 0, \\
\textbf{a/bBB}: & P - \varrho(Q + b\,V) = 0, \qquad V + \varrho(-U + b\,P) > 0,
\end{array} \tag{61}
$$

where $\varrho \equiv \frac{R_2}{R_1}$ is the complex structure modulus on the first two-torus.

The orientifold projection for all six inequivalent choices of lattice orientations is given in table 12.

Orientifold projection on bulk 3-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$				
$3-\text{cycle}$	ρ_1	ρ_2	ρ_3	ρ_4
a/bAA	$\rho_1 + (2b)\rho_3$	ρ_2	$-\rho_3$	$-\rho_4 + (2b)\rho_2$
a/bAB	$\rho_4 - (2b)\rho_2$	ρ_3	ρ_2	$\rho_1 + (2b)\rho_3$
a/bBB	$-\rho_1 - (2b)\rho_3$	$-\rho_2$	ρ_3	$\rho_4 - (2b)\rho_2$

Table 12: The orientifold projection on bulk three-cycles on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ for the six inequivalent lattices.

Representants of the four O6-plane orbits are given in table 13, and the resulting bulk wrapping numbers of the O6-planes are listed in table 14.

The charge assignment condition on O6-planes (14) can be written as

$$
\eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}}\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}} = \eta, \tag{62}
$$

where, e.g., $\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}$ is the charge of the orbit of O6-planes consisting of $\Omega\mathcal{R}\omega$ and $\Omega\mathcal{R}\omega^3$.

Torus wrapping numbers of O6-planes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$				
O6-plane orbit	$\frac{\text{angle}}{\pi}$ w.r.t. $\Omega\mathcal{R}$	**a/bAA**	**a/bAB**	**a/bBB**
		$(n^1, m^1; n^2, m^2; n^3, m^3)$ for one representant		
$\Omega\mathcal{R}$	$(0, 0, 0)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, 0; 1, 0)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, 0; 1, 1)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, 1; 1, 1)$
$\Omega\mathcal{R}\mathbb{Z}_4^{(1)}$	$(0, -\frac{1}{4}, \frac{1}{4})$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, -1; 1, 1)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, -1; 0, 1)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1, 0; 0, 1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0, 1; 0, -1; 1, 0)$	$(0, 1; 0, -1; 1, 1)$	$(0, 1; 1, -1; 1, 1)$
$\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}$	$(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$	$(0, -1; 1, 1; 1, 1)$	$(0, -1; 1, 1; 0, 1)$	$(0, -1; 0, 1; 0, 1)$

Table 13: Torus wrapping numbers for one representant per O6-plane orbit on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$. The corresponding bulk wrapping numbers per orbit are given in table 14.

Bulk wrapping numbers for the O6-planes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$															
	a/bAA					**a/bAB**					**a/bBB**				
orbit	P	Q	U	V	N_{O6}	P	Q	U	V	N_{O6}	P	Q	U	V	N_{O6}
$\Omega\mathcal{R}$	$\frac{1}{1-b}$	0	$\frac{b}{1-b}$	0	$8(1-b)$	$\frac{1}{1-b}$	$\frac{-b}{1-b}$	$\frac{b}{1-b}$	$\frac{1}{1-b}$	$4(1-b)$	0	$\frac{-2b}{1-b}$	0	$\frac{2}{1-b}$	$2(1-b)$
$\Omega\mathcal{R}\mathbb{Z}_4^{(1)}$	$\frac{2}{1-b}$	0	$\frac{2b}{1-b}$	0	$2(1-b)$	$\frac{1}{1-b}$	$\frac{-b}{1-b}$	$\frac{b}{1-b}$	$\frac{1}{1-b}$	$4(1-b)$	0	$\frac{-b}{1-b}$	0	$\frac{1}{1-b}$	$8(1-b)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	0	-1	0	0	$8(1-b)$	0	-1	-1	0	$4(1-b)$	0	0	-2	0	$2(1-b)$
$\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}$	0	-2	0	0	$2(1-b)$	0	-1	-1	0	$4(1-b)$	0	0	-1	0	$8(1-b)$

Table 14: Bulk wrapping numbers of the O6-plane orbits on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ computed from the torus wrapping numbers in table 13 using equation (60).

As stated in section 2.1.2, in the absence of torsion the number of exotic O6-plane orbits must be even, whereas in the presence of discrete torsion, an odd number needs to be exotic.

Using the orientifold images of three-cycles wrapped by D6-branes in table 12 and the

O6-planes in table 14, the RR tadpole cancellation conditions on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ read

$$
\begin{aligned}
\mathbf{a/bAA} \quad & \left[\sum_a N_a\, P_a - \left(8\,\eta_{\Omega\mathcal{R}} + 4\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}\right)\right](\rho_1 + b\rho_3) \\
& \qquad + \left[\sum_a N_a\,(Q_a + bV_a) + (1-b)\left(8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 4\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}}\right)\right]\rho_2 = 0, \\
\mathbf{a/bAB} \quad & \left[\sum_a N_a\,(P_a + V_a) - \left(8\,\eta_{\Omega\mathcal{R}} + 8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}\right)\right](\rho_1 + \rho_4 + b(\rho_3 - \rho_2)) \\
+ & \left[\sum_a N_a\,(Q_a + U_a - b(P_a - V_a)) + (1-b)\left(8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}}\right)\right](\rho_2 + \rho_3) = 0, \\
\mathbf{a/bBB} \quad & \left[\sum_a N_a\, V_a - \left(4\,\eta_{\Omega\mathcal{R}} + 8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}\right)\right](\rho_4 - b\rho_2) \\
& \qquad + \left[\sum_a N_a\,(U_a - bP_a) + (1-b)\left(4\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 8\,\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}}\right)\right]\rho_3 = 0,
\end{aligned}
\tag{63}
$$

for both choices of discrete torsion $\eta = \pm 1$. These are per lattice $2 = 1 + h_{21}^U$ independent conditions.

Next, we list how redundancies due to symmetires are removed in the counting of inequivalent models:

- The $\mathbb{Z}_4$ projection (59) on $T^2_{(3)}$ exchanges the torus wrapping numbers $(n^3, m^3) =$ (odd,even) $\leftrightarrow$ (even,odd) and preserves (odd,odd). On can therefore choose $\boxed{n^3 = \text{odd}}$. This singles out one orbifold image for m^3 even, but the redundancy for (n^3, m^3)=(odd,odd) survives.
- The orientation of the one-cycle on $T^2_{(3)}$ is fixed by requiring $\boxed{n^3 > 0}$.
- Avoiding simultaneous sign flips on $T^2_{(1)} \times T^2_{(2)}$ and no double-counting of the orientifold image is achieved by requiring $0 \leq \pi\phi^{(1)} \leq \frac{\pi}{2}$ in equation (32): $\boxed{m^1 + b\,n^1 \geq 0 \text{ and } n^1 \geq 0}$. These conditions are sufficient for non-trivial angles.
- For $\pi\phi^{(1)} \in \{0, \frac{\pi}{2}\}$ one has to impose an additional condition on the third two-torus, e.g. by demanding $\boxed{m^3 \leq 0 \text{ on } \mathbf{A} \text{ and } |m^3| \leq n^3 \text{ on } \mathbf{B} \text{ for } T^2_{(3)}}$.
  The limiting cases $\pi\phi^{(3)} \in \{0, -\frac{\pi}{2}\}$ correspond to three-cycles parallel to the O6-planes.

With these constraints on torus wrapping numbers, a classification of supersymmetric three-cycles can be performed in an economic way.

4.2 A discussion on three generations on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ without and with discrete torsion

Since the bulk three-cycles are identical for both choices of torsion, the no-go theorem on three generations found in [67] is expected to still be valid, as we discuss below for several scenarios.

1. The number of chiral bifundamental representations of the QCD stack with a second stack is given by

$$\Pi_a \circ (\Pi_b + \Pi_{b'}) = \begin{cases} 2U_aP_b + 2V_aQ_b + 2b(V_aV_b - P_aP_b) & \mathbf{a/bAA} \\ (V_a - P_a)(Q_b + U_b) + (U_a - Q_a)(P_b + V_b) \\ -2b(P_aV_b + V_aP_b) & \mathbf{a/bAB} \\ -2Q_aV_b - 2P_aU_b + 2b(P_aP_b - V_aV_b) & \mathbf{a/bBB} \end{cases} \tag{64}$$

   On the **aAA** and **aBB** lattices, it is obvious that only an even number of generations can occur. This leaves at first glance the **bAA**, **bBB** and **a/bAB** lattices as potentially interesting backgrounds for model building.

2. The number of chiral symmetrics on the QCD stack is given by

$$\chi^{\mathbf{Sym}_a} = \frac{1}{2}\left(\Pi_a \circ \Pi_{a'} - \Pi_a \circ \Pi_{O6}\right) \tag{65}$$

   with

$$\Pi_a \circ \Pi_{a'} = \begin{cases} 2P_aU_a + 2V_aQ_a + 2b(V_a^2 - P_a^2) & \mathbf{a/bAA} \\ 2U_aV_a - 2P_aQ_a - 4bP_aV_a & \mathbf{a/bAB} \\ -2P_aU_a - 2V_aQ_a - 2b(V_a^2 - P_a^2) & \mathbf{a/bBB} \end{cases},$$

$$\Pi_a \circ \Pi_{O6} = \begin{cases} (4\eta_{\Omega\mathcal{R}} + 2\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}})(U_a - bP_a) \\ -(4\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 2\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}\mathbb{Z}_4^{(1)}})(1-b)V_a & \mathbf{a/bAA} \\ (4\eta_{\Omega\mathcal{R}} + 4\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}})(U_a - Q_a - b(P_a + V_a)) \\ +(4\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 4\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}\mathbb{Z}_4^{(1)}})(1-b)(P_a - V_a) & \mathbf{a/bAB} \\ -(2\eta_{\Omega\mathcal{R}} + 4\eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}})(Q_a + bV_a) \\ +(2\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} + 4\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}\mathbb{Z}_4^{(1)}})(1-b)P_a & \mathbf{a/bBB} \end{cases}, \tag{66}$$

and the number of chiral antisymmetrics is computed from

$$\chi^{\mathbf{Anti}_a} = \frac{1}{2}\left(\Pi_a \circ \Pi_{a'} + \Pi_a \circ \Pi_{O6}\right) \stackrel{\chi^{\mathbf{Sym}_a}=0}{=} \Pi_a \circ \Pi_{a'}, \tag{67}$$

where in the second step we inserted the requirement of no chiral symmetric representation for phenomenological reasons. Furthermore,

$$0 \leq |\chi^{\mathbf{Anti}_a}| \leq 3 \tag{68}$$

ensures the absence of an excess of right-handed quarks, where $\chi^{\mathbf{Anti}_a} = 0$ is required for left-right symmetric and Pati-Salam models and $|\chi^{\mathbf{Anti}_a}| = 3$ for $SU(5)$ GUTs. Formula (66) reveals, that an odd number of antisymmetric generations is only possible on the **bAA** and **bBB** lattices. This means that $SU(5)$ GUTs with three generations and no chiral exotics are excluded on all other lattices.

3. Supersymmetry ensures that there are only two choices, $(n^1_a, m^1_a) \in \{(\frac{1}{1-b}, \frac{-b}{1-b}), (0,1)\}$, for which

$$\chi^{\mathbf{Anti}_a} = \chi^{\mathbf{Sym}_a} = 0. \tag{69}$$

Supersymmetry on the D6$_a$-brane with these special values for (n^1_a, m^1_a) also renders the number of bifundamental chiral generations with an arbitrary (supersymmetric or non-supersymmetric) D6$_b$-brane zero for any lattice, see equation (64). This implies that supersymmetric left-right symmetric or Pati-Salam models are excluded on any lattice.

The discussion above is independent of the choice of discrete torsion or an exotic O6-plane. We expect that a thorough case by case study of the remaining possibilities with some right-handed quarks in the antisymmetric representation will rule out model building on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ completey.

4.3 The K-theory constraint on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$

The K-theory constraint is for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ independent of the choice of discrete torsion. There exist four possible probe D6-branes per lattice, namely those parallel to the O6-planes with bulk wrapping numbers given in table 14, which provide for at most two independent K-theory conditions. Computing the intersection numbers (27) and using the RR tadpole cancellation conditions (63) to simplify the K-theory constraint, one ends up with

$$\begin{array}{llcl} \mathbf{aAA}: & \sum_a N_a U_a & = 0 \bmod 2 = & \sum_a N_a V_a, \\ \mathbf{aAB}: & \sum_a N_a (Q_a - U_a) & = 0 \bmod 2 = & \sum_a N_a (P_a - V_a), \\ \mathbf{aBB}: & \sum_a N_a Q_a & = 0 \bmod 2 = & \sum_a N_a P_a, \end{array} \tag{70}$$

and all constraints trivially fulfilled on the **b** type torus $T^2_{(1)}$. These conditions may be too restrictive, if some of the probe branes had $SO(2M)$ instead of $Sp(2M)$ gauge groups. We will discuss this subtlety further in appendix B.2, and in section 4.5 we give an example where an orientifold invariant D6-brane indeed supports an $SO(2M)$ gauge factor.

4.4 Exeptional three-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ with discrete torsion

The exceptional three-cycles in the $\mathbb{Z}_4$ twisted sectors for the case of discrete torsion arise as follows

$$\begin{aligned} &\delta_l^{(k)} = 2\,\tilde{d}_{ij}^{(k)} \otimes \pi_1, \qquad \tilde{\delta}_l^{(k)} = 2\,\tilde{d}_{ij}^{(k)} \otimes \pi_2 \quad l \in \{1 \ldots 4\} \;\Leftrightarrow\; (ij) \in \{(11),(12),(21),(22)\}, \qquad k \in \{1,2\} \\ &\delta_l^{(k)} \circ \tilde{\delta}_{l'}^{(k')} \sim \; \delta^{kk'} \delta_{ll'} \end{aligned} \tag{71}$$

with the intersection form of two-cycles inherited from $T^4/\mathbb{Z}_4$,

$$d_{ij}^{(k)} \circ d_{i'j'}^{(k')} = \delta_{ii'}\delta_{jj'} \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}_{kk'}, \tag{72}$$

where $d^{(k)}$ runs over the two exceptional cycles $\tilde{d}^{(k)}$ associated to $\mathbb{Z}_4$ twisted sectors and on exceptional cycle belonging to the $\mathbb{Z}_2$ twisted sector at the same fixed point. The orientifold projection on the twisted three-cycles is on this orbifold independent of the lattice orientation as displayed in table 15.

$\Omega\mathcal{R}$ **projection on exceptional 3-cycles on** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$		
$3-$ cycle	$\delta_l^{(k)}$	$\tilde{\delta}_l^{(k)}$
a/bAA, **a/bAB**, **a/bBB**	$-\eta_{\mathbb{Z}_4^{(1)}}\,\delta_l^{(k)}$	$\eta_{\mathbb{Z}_4^{(1)}}\,\tilde{\delta}_l^{(k)}$

Table 15: The orientifold projection on twisted three-cycles on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times \Omega\mathcal{R})$ with discrete torsion. The prefactor $\eta_{\mathbb{Z}_4^{(1)}} \equiv \eta_{\Omega\mathcal{R}} \cdot \eta_{\Omega\mathcal{R}\mathbb{Z}_4^{(1)}}$ depends on the choice of the exotic O6-plane, cf. eq. (14).

4.5 A $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ example with discrete torsion

In [67], a supersymmetric model on the **bBB** lattice on $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ without discrete torsion was presented. We list the torus and bulk wrapping numbers of the D6-branes in that model in table 16 after choosing each orbifold and orientifold representant as discussed at the end of section 4.1.

Supersymmetric bulk 3-cycles on the bBB lattice on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$						
brane/orbit	$\frac{\text{angle}}{\pi}$	$(n^1,m^1;n^2,m^2;n^3,m^3)$	P	Q	U	V
a_1	$(\frac{1}{4},-\frac{1}{4},0)$	$(1,0;1,0;1,1)$	1	0	0	1
a_2	$(\frac{1}{4},0,-\frac{1}{4})$	$(1,0;1,1;1,0)$	1	0	0	1
$b_1/\Omega\mathcal{R}\mathbb{Z}_4^{(1)}$	$(0,\frac{1}{4},-\frac{1}{4})$	$(2,-1;0,1;1,0)$	0	-1	0	2
$b_2/\Omega\mathcal{R}$	$(0,0,0)$	$(2,-1;1,1;1,1)$	0	-2	0	4
$c_1/\Omega\mathcal{R}\mathbb{Z}_4^{(1)}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2},-\frac{1}{4},-\frac{1}{4})$	$(0,1;1,0;1,0)$	0	0	-1	0
$c_2/\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2},0,-\frac{1}{2})$	$(0,1;1,1;1,-1)$	0	0	-2	0

(73)

Table 16: Some examples of supersymmetric three-cycles on the **bBB** lattice on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_4\times\Omega\mathcal{R})$ with complex structure modulus $\varrho=2$. In [67], RR tadpole cancellation was achieved by setting $N_{a_i}=3$ and $N_{b_i}=N_{c_i}=1$ for $i=1,2$.

The RR tadpole cancellation conditions (63) on **bBB** with the choice of an exotic O6-plane $\eta_{\Omega\mathcal{R}}=-1$ read

$$\begin{aligned}\sum_a N_a\, V_a =&4,\\ \sum_a N_a\left(U_a-\frac{1}{2}\,P_a\right)=&-6.\end{aligned}\tag{74}$$

A supersymmetric solution is given for $N_{a_1}=4$ and $N_{c_2}=2$ with the cycles as defined in table 16, i.e.

$$\begin{aligned}\Pi_{a_1}&=\frac{1}{2}\left(\rho_1+\rho_4\right),\\ \Pi_{c_2}&=-\rho_2.\end{aligned}\tag{75}$$

The gauge group is $U(4)_{a_1}\times SO(4)_{c_2}$, where the diagonal $U(1)\subset U(4)_{a_1}$ is anomalous and

massive. The relevant intersection numbers are

$$\begin{aligned}\Pi_{a_1} \circ \Pi_{c_2} &= -2,\\ \Pi_{a_1} \circ \Pi_{a_1'} &= 0,\\ \Pi_{a_1} \circ \Pi_{O6} &= 2,\end{aligned} \tag{76}$$

from which the chiral part of the spectrum is derived,

$$(\mathbf{6}_{a_1}, \mathbf{1}) + (\overline{\mathbf{10}}_{a_1}, \mathbf{1}) + 2\,(\overline{\mathbf{4}}_{a_1}, \mathbf{4}_{c_2}). \tag{77}$$

Using the computational methods described in appendix B, the massless spectrum contains also the following non-chiral matter,

- three multiplets in the adjoint representation $(\mathbf{16}_{a_1}, \mathbf{1})$ of $U(4)_{a_1}$ from the a_1a_1 sector plus two further adjoints at orbifold intersections $a_1(\omega a_1)$,
- one non-chiral pair of bifundamentals $[(\mathbf{4}_{a_1}, \mathbf{4}_{c_2}) + c.c.]$ under $U(4)_{a_1} \times SO(4)_{c_2}$,
- two non-chiral pairs of antisymmetrics $2 \times [(\mathbf{6}_{a_1}, \mathbf{1}) + c.c.]$ of $U(4)_{a_1}$,
- two multiplets in the antisymmetric and one in the symmetric representation of $SO(4)_{c_2}$ from the c_2c_2 sector plus four more antisymmetrics from the $c_2(\omega c_2)$ sector.

The K-theory constraint is trivially fulfilled on the **bBB** lattice, and also due to the construction with gauge groups of even rank only.

5 The $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ orientifolds

The orbifold shifts for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ are given by

$$\vec{v} = \frac{1}{2}(1, -1, 0), \quad \vec{w} = \frac{1}{6}(0, 1, -1). \tag{78}$$

There are two sub-sectors, $\vec{v} + \vec{w} = \frac{1}{6}(3, -2, -1)$ and $\vec{v} + 2\vec{w} = \frac{1}{6}(3, -1, -2)$, which are the generators of two different $T^6/\mathbb{Z}_6'$ orbifolds. As will be discussed in detail below, the compactification lattices and bulk cycles of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_6'$ are identical up to normalisation. Also two of the three $\mathbb{Z}_2^{(i)}$ twisted sectors of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ with discrete torsion are inherited from the two $T^6/\mathbb{Z}_6'$ subsectors, as discussed in detail in table 62 in appendix D. This correspondence allows for using partial results, e.g. the classification of supersymmetric bulk three-cycles, from the well-studied $T^6/\mathbb{Z}_6'$ orbifold [68, 69] in order to

construct D6-brane models on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ orbifold with discrete torsion. It also gives reason to hope that on this product orbifold, new phenomenologically appealing spectra with three Standard Model generations will be found.

Choosing the factorisable torus lattice $SU(2)^2 \times SU(3)^2$ leads to the Hodge numbers for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ without and with discrete torsion listed in table 4 and the decomposition into $h_{11}^{\pm}$ upon orientifolding in table 46. The three-cycles, on which D6-branes can be wrapped, are counted by the following Hodge numbers

$$\eta = 1: \; h_{21} = 1_{\text{bulk}} + 2_{\mathbb{Z}_3}, \qquad \eta = -1: \; h_{21} = 1_{\text{bulk}} + 14_{\mathbb{Z}_2} + 2_{\mathbb{Z}_6} + 2_{\mathbb{Z}_3}, \tag{79}$$

where only the bulk and $\mathbb{Z}_2$ twisted cycles have an interpretation in terms of open string loop amplitudes as discussed in section 2.

In the absence of discrete torsion, four independent bulk three-cycles can be used for model building, where supersymmetry projects out a one-dimensional sub-space, but leaves enough freedom for engineering chirality.

5.1 The bulk part

A basis of bulk three-cycles is given by

$$\begin{aligned}
\rho_1 &\equiv 4 \sum_{i=0}^{2} \theta^i(\pi_{135}) = 4\,(\pi_{135} + \pi_{145} - 2\,\pi_{146} + \pi_{136}), \\
\rho_2 &\equiv 4 \sum_{i=0}^{2} \theta^i(\pi_{136}) = 4\,(2\,\pi_{136} + 2\,\pi_{145} - \pi_{146} - \pi_{135}), \\
\rho_3 &\equiv 4 \sum_{i=0}^{2} \theta^i(\pi_{235}) = 4\,(\pi_{235} + \pi_{245} - 2\,\pi_{246} + \pi_{236}), \\
\rho_4 &\equiv 4 \sum_{i=0}^{2} \theta^i(\pi_{236}) = 4\,(2\,\pi_{236} + 2\,\pi_{245} - \pi_{246} - \pi_{235}),
\end{aligned} \tag{80}$$

with non-vanishing intersection numbers

$$\begin{aligned}
\rho_1 \circ \rho_3 &= \rho_2 \circ \rho_4 = 8, \\
\rho_1 \circ \rho_4 &= \rho_2 \circ \rho_3 = 4.
\end{aligned} \tag{81}$$

The wrapping numbers of a factorisable three-cycle on the underlying torus lattice transform as

$$\begin{pmatrix} n^1 & m^1 \\ n^2 & m^2 \\ n^3 & m^3 \end{pmatrix} \stackrel{\mathbb{Z}_6^{(1)}}{\to} \begin{pmatrix} n^1 & m^1 \\ -m^2 & n^2 + m^2 \\ n^3 + m^3 & -n^3 \end{pmatrix} \stackrel{\mathbb{Z}_6^{(1)}}{\to} \begin{pmatrix} n^1 & m^1 \\ -(n^2 + m^2) & n^2 \\ m^3 & -(n^3 + m^3) \end{pmatrix} \tag{82}$$

under the $\mathbb{Z}_6$ symmetry. A bulk three-cycle on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ orbifold takes thus the form

$$\Pi^{\text{bulk}} = P\ \rho_1 + Q\ \rho_2 + U\ \rho_3 + V\ \rho_4 \tag{83}$$

with the bulk wrapping numbers

$$\begin{aligned} P \equiv & n^1 X, \quad Q \equiv n^1 Y, \quad U \equiv m^1 X, \quad V \equiv m^1 Y, \\ \text{with} \quad & X \equiv n^2 n^3 - m^2 m^3, \quad Y \equiv n^2 m^3 + m^2 n^3 + m^2 m^3, \end{aligned} \tag{84}$$

and the intersection number of two bulk cycles is given by

$$\Pi_a^{\text{bulk}} \circ \Pi_b^{\text{bulk}} = 4 \left\{ 2\left(P_a U_b - P_b U_a + Q_a V_b - Q_b V_a\right) + P_a V_b - P_b V_a + Q_a U_b - Q_b U_a \right\}. \tag{85}$$

The cycles $\frac{1}{2}\Pi^{\text{bulk}}$ thus all have integer intersection numbers among each other. They do, however, not form an unimodular basis even in the absence of discrete torsion since there are always three-cycles from the $\mathbb{Z}_3^{(1)}$-twisted sector present as briefly discussed below in section 5.4.

The orientifold projection on the bulk cycles is independent of the choice of discrete torsion and the exotic O6-plane and given for all six inequivalent choices of lattice orientations in table 17.

The orientifold projection on bulk 3-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$				
$3-\text{cycle}$	ρ_1	ρ_2	ρ_3	ρ_4
a/bAA	$\rho_1 - (2b)\rho_3$	$\rho_1 - \rho_2 - (2b)[\rho_3 - \rho_4]$	$-\rho_3$	$\rho_4 - \rho_3$
a/bAB	$\rho_2 - (2b)\rho_4$	$\rho_1 - (2b)\rho_3$	$-\rho_4$	$-\rho_3$
a/bBB	$\rho_2 - \rho_1 - (2b)[\rho_4 - \rho_3]$	$\rho_2 - (2b)\rho_4$	$\rho_3 - \rho_4$	$-\rho_4$

Table 17: The orientifold projection on bulk three-cycles on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ without and with discrete torsion. $b = 0, 1/2$ labels the untilted and tilted two-torus $T^2_{(1)}$.

The supersymmetry conditions (29) on bulk three-cycles with the explicit orbifold expressions (30) and (31) read

$$\begin{aligned} \textbf{a/bAA}: &\quad 3\,Q + \varrho[2U + V + b\,(2P + Q)] = 0, &\quad 2\,P + Q - \varrho[V + b\,Q] > 0, \\ \textbf{a/bAB}: &\quad Q - P + \varrho[U + V + b\,(P + Q)] = 0, &\quad 3\,(P + Q) + \varrho[U - V + b\,(P - Q)] > 0, \\ \textbf{a/bBB}: &\quad -3\,P + \varrho[U + 2\,V + b\,(P + 2\,Q)] = 0, &\quad P + 2\,Q + \varrho(U + b\,P) > 0, \end{aligned} \tag{86}$$

in terms of the bulk wrapping numbers and the complex structure modulus $\varrho \equiv \sqrt{3}\frac{R_2}{R_1}$ on the first two-torus.[5]

The orientifold planes lie in four orbits, $\Omega\mathcal{R}\omega^{2l}$, $\Omega\mathcal{R}\theta$, $\Omega\mathcal{R}\omega^{2l+1}$ and $\Omega\mathcal{R}\theta\omega^{2l+1}$ under the $\mathbb{Z}_6^{(1)}$ orbifold generator; we label these orbits in the following by one of their representants namely, $\Omega\mathcal{R}$, $\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$, $\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$ and $\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$, respectively. The torus wrapping numbers for one representant of each orbit are given in table 18 for all inequivalent lattices, and in table 19 the corresponding bulk wrapping numbers using the definition (84) are listed.

Torus wrapping numbers for the four O6-plane orbits on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$				
O6-plane	$\frac{\text{angle}}{\pi}$	**a/bAA**	**a/bAB**	**a/bBB**
		$(n^1, m^1; n^2, m^2; n^3, m^3)$		
$\Omega\mathcal{R}$	$(0,0,0)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1,0; 1,0)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1,0; 1,1)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; 1,1; 1,1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$(0, \frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{1-b}, \frac{-b}{1-b}; -1,2; 1,-2)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; -1,2; 1,-1)$	$(\frac{1}{1-b}, \frac{-b}{1-b}; -1,1; 1,-1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0,1; 1,-2; 1,0)$	$(0,1; 1,-2; 1,1)$	$(0,1; 1,-1; 1,1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0,1; 1,0; 1,-2)$	$(0,1; 1,0; 1,-1)$	$(0,1; 1,1; 1,-1)$

Table 18: The torus wrapping numbers for one representant of each O6-plane orbit on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$. The angle w.r.t. the $\Omega\mathcal{R}$ invariant plane is listed in the second column. The other two torus cycles per orbit can be computed by using (82).

The charge assignment condition (14) on exotic O6-planes reads the same as for $\mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\eta_{\Omega\mathcal{R}} \prod_{k=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}} = \eta. \tag{87}$$

This is due to the fact that the four O6-plane orbits split exactly into the orbits of $\Omega\mathcal{R}$ and $\Omega\mathcal{R}\mathbb{Z}_2^{(k)}$ with $k = 1, 2, 3$ as discussed above.

The orientifold projection on bulk 3-cycles in table 17 and the O6-plane wrapping numbers

[5] The supersymmetry conditions agree with those on $T^6/\mathbb{Z}_6'$ in [68, 69, 79] up to permutation of the two-tori and with the change in the definition of the complex structure modulus by a factor of two.

Bulk wrapping numbers for the four O6-plane orbits on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$												
	a/bAA				**a/bAB**				**a/bBB**			
orbit	P	Q	U	V	P	Q	U	V	P	Q	U	V
$\Omega\mathcal{R}$	$\frac{1}{1-b}$	0	$\frac{-b}{1-b}$	0	$\frac{1}{1-b}$	$\frac{1}{1-b}$	$\frac{-b}{1-b}$	$\frac{-b}{1-b}$	0	$\frac{3}{1-b}$	0	$\frac{-3\,b}{1-b}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$\frac{3}{1-b}$	0	$\frac{-3\,b}{1-b}$	0	$\frac{1}{1-b}$	$\frac{1}{1-b}$	$\frac{-b}{1-b}$	$\frac{-b}{1-b}$	0	$\frac{1}{1-b}$	0	$\frac{-b}{1-b}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	0	0	1	-2	0	0	3	-3	0	0	2	-1
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	0	0	1	-2	0	0	1	-1	0	0	2	-1

Table 19: Bulk wrapping numbers for the O6-plane orbits on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$. The number of identical O6-planes is $N_{O6} = 2(1-b)$ with $b = 0, 1/2$ parameterising the shape on the first two-torus.

in table 19 lead to the bulk RR tadpole cancellation conditions for all six lattices,

$$\begin{aligned}
&\mathbf{a/bAA}: \quad \left[\sum_a N_a\,(2\,P_a + Q_a) - 2^{\frac{5-\eta}{2}}\,\left(\eta_{\Omega\mathcal{R}} + 3\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}\right)\right](\rho_1 - b\rho_3) = \\
&- \left[\sum_a N_a\,(V_a + bQ_a) + 2^{\frac{5-\eta}{2}}\,(1-b)\,\left(\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}}\right)\right](-\rho_3 + 2\,\rho_4), \\
&\mathbf{a/bAB}: \quad \left[\sum_a N_a\,(P_a + Q_a) - 2^{\frac{5-\eta}{2}}\,\left(\eta_{\Omega\mathcal{R}} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}\right)\right](\rho_1 + \rho_2 - b(\rho_3 + \rho_4)) = \\
&- \left[\sum_a N_a\,(U_a - V_a + b(P_a - Q_a)) - 2^{\frac{5-\eta}{2}}\,(1-b)\,\left(\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}} + 3\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}}\right)\right](\rho_3 - \rho_4), \\
&\mathbf{a/bBB}: \quad \left[\sum_a N_a\,(P_a + 2\,Q_a) - 2^{\frac{5-\eta}{2}}\,\left(3\,\eta_{\Omega\mathcal{R}} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}\right)\right](\rho_2 - b\rho_4) = \\
&- \left[\sum_a N_a\,(U_a + bP_a) - 2^{\frac{5-\eta}{2}}\,(1-b)\,\left(\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}}\right)\right](2\rho_3 - \rho_4).
\end{aligned} \tag{88}$$

Per lattice, two independent bulk RR tadpole cancellation conditions arise which depend on the choice of discrete torsion in two ways: $2^{\frac{\eta-1}{2}}$ parameterises the fact that without discrete torsion, the D6-branes wrap cycles $\frac{1}{2}\Pi^{\text{bulk}}$, whereas with discrete torsion they wrap $\frac{1}{4}\left(\Pi^{\text{bulk}} + \sum^i \Pi^{\mathbb{Z}_2^{(i)}}\right)$, while in both cases the non-dynamical O6-planes span $\frac{1}{4}\Pi^{\text{bulk}}$.

Furthermore, in the absence of discrete torsion $\eta_{\Omega\mathcal{R}} = \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} = 1$ for all $i = 1, 2, 3$, whereas with discrete torsion, one O6-plane has to be chosen exotic.

Before discussing the exceptional three-cycles, let us discuss how a multiple counting of equivalent models is avoided. The following conditions can be imposed:

- $\boxed{(n^3, m^3) = \text{(odd,odd)}}$ selects the representation through a specific $\mathbb{Z}_6$ orbifold image,
- $\boxed{n^3 > 0}$ fixes the orientation on the last two-torus $T^2_{(3)}$,
- $\boxed{m^1 + b\, n^1 \geq 0 \text{ and } n^1 > 0}$ or $\boxed{(n^1, m^1) = (0, 1)}$ fixes the orientation on $T^2_{(1)}$ and selects the orientifold image with angle $0 \leq \pi\phi^{(1)} \leq \frac{1}{2}$,
- $\boxed{\text{for } (n^1, m^1) \in \{(\frac{1}{1-b}, \frac{-b}{1-b}), (0, 1)\}}$, the orientifold images are not yet distinguished, in this case impose as additional condition on $T^2_{(3)}$ that $-\frac{\pi}{2} < \pi\phi^{(3)} < 0$, which in terms of the wrapping numbers reads $\boxed{-2n^3 < m^3 < 0 \text{ on } \mathbf{A} \text{ and } |m^3| < n^3 \text{ on } \mathbf{B}}$.
- D6-branes parallel to some $\Omega\mathcal{R}\theta^n\omega^m$ invariant plane are treated separately. The torus wrapping numbers of one representant per orbit are listed in table 18.

These conditions simplify the systematic analysis of fractional supersymmetric three-cycles in the case with discrete torsion significantly, since the assignments of exceptional three-cycles in tables 52 and 53 only need to be considered for $(n^3, m^3) = \text{(odd,odd)}$.

5.2 The $\mathbb{Z}_2$ twisted parts

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ orbifold with discrete torsion has three sectors of exceptional three-cycles at $\mathbb{Z}_2^{(i)}$ fixed points with $i = 1, 2, 3$ labelling the two-torus which is left invariant. These exceptional three-cycles are:

1. in the $\mathbb{Z}_2^{(1)}$ sector: twelve independent three-cycles

$$
\begin{aligned}
\varepsilon_1^{(1)} &= 6\, e_{11}^{(1)} \otimes \pi_1, \qquad\qquad , & \tilde{\varepsilon}_1^{(1)} &= 6\, e_{11}^{(1)} \otimes \pi_2, \\
\varepsilon_1^{(1)} &= 2\left(e_{41}^{(1)} + e_{51}^{(1)} + e_{61}^{(1)}\right) \otimes \pi_1, & \tilde{\varepsilon}_1^{(1)} &= 2\left(e_{41}^{(1)} + e_{51}^{(1)} + e_{61}^{(1)}\right) \otimes \pi_2, \\
\varepsilon_2^{(1)} &= 2\left(e_{14}^{(1)} + e_{15}^{(1)} + e_{16}^{(1)}\right) \otimes \pi_1, & \tilde{\varepsilon}_2^{(1)} &= 2\left(e_{14}^{(1)} + e_{15}^{(1)} + e_{16}^{(1)}\right) \otimes \pi_2, \\
\varepsilon_3^{(1)} &= 2\left(e_{44}^{(1)} + e_{56}^{(1)} + e_{65}^{(1)}\right) \otimes \pi_1, & \tilde{\varepsilon}_3^{(1)} &= 2\left(e_{44}^{(1)} + e_{56}^{(1)} + e_{65}^{(1)}\right) \otimes \pi_2, \\
\varepsilon_4^{(1)} &= 2\left(e_{45}^{(1)} + e_{54}^{(1)} + e_{66}^{(1)}\right) \otimes \pi_1, & \tilde{\varepsilon}_4^{(1)} &= 2\left(e_{45}^{(1)} + e_{54}^{(1)} + e_{66}^{(1)}\right) \otimes \pi_2, \\
\varepsilon_5^{(1)} &= 2\left(e_{46}^{(1)} + e_{55}^{(1)} + e_{64}^{(1)}\right) \otimes \pi_1, & \tilde{\varepsilon}_5^{(1)} &= 2\left(e_{46}^{(1)} + e_{55}^{(1)} + e_{64}^{(1)}\right) \otimes \pi_2,
\end{aligned}
\tag{89}
$$

   with intersection form

$$
\begin{aligned}
\varepsilon_0^{(1)} \circ \tilde{\varepsilon}_0^{(1)} &= -12, \\
\varepsilon_i^{(1)} \circ \tilde{\varepsilon}_j^{(1)} &= -4\, \delta_{ij} \qquad \text{for } i, j \in \{1 \ldots 5\},
\end{aligned}
\tag{90}
$$

   and all other intersections vanishing.

2. in the $\mathbb{Z}_2^{(\alpha)}$ sector with $\alpha \in \{2, 3\}$, we label the fixed points on $T^2_{(1)}$ by $k = 1, 2, 3, 4$ and define the following exceptional three-cycles

$$
\begin{aligned}
\varepsilon_k^{(\alpha)} &= 2\, \left(e_{k4}^{(\alpha)} \otimes \pi_3 + e_{k6}^{(\alpha)} \otimes \pi_{-4} + e_{k5}^{(\alpha)} \otimes \pi_{4-3}\right), \\
\tilde{\varepsilon}_k^{(\alpha)} &= 2\, \left(e_{k4}^{(\alpha)} \otimes \pi_4 + e_{k6}^{(\alpha)} \otimes \pi_{3-4} + e_{k5}^{(\alpha)} \otimes \pi_{-3}\right),
\end{aligned}
\tag{91}
$$

   with intersection form

$$
\varepsilon_k^{(\alpha)} \circ \tilde{\varepsilon}_l^{(\alpha)} = -4\, \delta_{kl}, \tag{92}
$$

   and all others vanishing. These two sectors are (up to normalisation and permutation of tori) equivalent to the exceptional three-cycles at $\mathbb{Z}_2$ fixed points on the $T^6/\mathbb{Z}_6'$ orbifold discussed in [68, 69].

Supersymmetric fractional three-cycles are composed of a bulk cycle satisfying (86) and those exceptional cycles whose fixed points are traversed by the bulk cycle for a given displacement $\vec{\sigma}$ on $T^2_{(1)} \times T^2_{(2)} \times T^2_{(3)}$. The assignemt of a single $\mathbb{Z}_2^{(i)}$ fixed point and its exceptional cycle is given in table 20. We relegate the detailed discussion on the composition of exceptional three-cycles in dependence of wrapping numbers, discrete Wilson lines and displacements to tables 52, 53 and 54 in appendix C, but notice here that only the configurations with (n^3, m^3) need to be evaluated explicitly.

The orientifold projection on exceptional three-cycles depends on the transformation of the $\mathbb{Z}_2^{(i)}$ fixed points under $\mathcal{R}$ and the sign factor (47) under Ω, that is $\eta_{(i)} \equiv \eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}$,

$\mathbb{Z}_2^{(1)}$ **fixed points and 3-cycles**	
f.p.$^{(1)} \otimes (n^1\pi_1 + m^1\pi_2)$	orbit
11	$n^1\varepsilon_0^{(1)} + m^1\tilde{\varepsilon}_0^{(1)}$
$41,\ 51,\ 61$	$n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}$
$14,\ 15,\ 16$	$n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}$
$44,\ 56,\ 65$	$n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}$
$45,\ 54,\ 66$	$n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}$
$46,\ 55,\ 64$	$n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}$

$\mathbb{Z}_2^{(2)}$ **and** $\mathbb{Z}_2^{(3)}$ **fixed points and 3-cycles**	
f.p.$^{(\alpha)} \otimes (n^\alpha\pi_{2\alpha-1} + m^\alpha\pi_{2\alpha})$	orbit
$k1$	$-$
$k4$	$n^\alpha\varepsilon_k^{(\alpha)} + m^\alpha\tilde{\varepsilon}_k^{(\alpha)}$
$k5$	$m^\alpha\varepsilon_k^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_k^{(\alpha)}$
$k6$	$-(n^\alpha + m^\alpha)\varepsilon_k^{(\alpha)} + n^\alpha\tilde{\varepsilon}_k^{(\alpha)}$

Table 20: Correspondence between $\mathbb{Z}_2^{(i)}$ fixed points and exceptional three-cycles for $i = 1$ on the left and $\alpha \in \{2,3\}$ on the right on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ orbifold.

$\Omega\mathcal{R}$ **on exceptional 3-cycles for** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$**, Part I**				
3-cycle	$\varepsilon_i^{(1)}$	$\tilde{\varepsilon}_i^{(1)}$	$i = i'$	$i \leftrightarrow i'$
a/bAA	$\eta_{(1)}\left(-\varepsilon_{i'}^{(1)} + (2b)\tilde{\varepsilon}_{i'}^{(1)}\right)$	$\eta_{(1)}\,\tilde{\varepsilon}_{i'}^{(1)}$	$0,1,2,3$	$4,5$
a/bAB			$0,1,2,5$	$3,4$
a/bBB			$0,1,2,4$	$3,5$

Table 21: Orientifold projection on exceptional three-cycles in the $\mathbb{Z}_2^{(1)}$ twisted sector on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ with discrete torsion. The sign factor $\eta_{(1)} \equiv \eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}$ depends on the choice of the exotic O6-plane.

which depends on the choice of the exotic O6-plane. The results for the i^{th} exceptional sector are given in table 21, 22 and 23 for $i = 1,2,3$, respectively.

A fractional three-cycle can be formally expanded as

$$\begin{aligned}\Pi^{\text{frac}} = \frac{1}{4}\Pi^{\text{bulk}} + \frac{1}{4}\sum_{\alpha=1}^{3}\Pi^{\mathbb{Z}_2^{(\alpha)}} = \quad & \frac{1}{4}\left(P\rho_1 + Q\rho_2 + U\rho_3 + V\rho_4\right) \\ & + \frac{1}{4}\left[\sum_{i=0}^{5}\left(x_i^{(1)}\,\varepsilon_i^{(1)} + y_i^{(1)}\,\tilde{\varepsilon}_i^{(1)}\right) + \sum_{\alpha=2,3}\sum_{i=1}^{4}\left(x_i^{(\alpha)}\,\varepsilon_i^{(\alpha)} + y_i^{(\alpha)}\,\tilde{\varepsilon}_i^{(\alpha)}\right)\right],\end{aligned} \tag{93}$$

where the integer coefficients $(x_i^{(\alpha)}, y_i^{(\alpha)})$ can be read off from tables 52 to 54 in appendix C.

$\Omega\mathcal{R}$ **on exceptional 3-cycles for** $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$**, Part II**				
3-cycle	$\varepsilon_i^{(2)}$	$\tilde{\varepsilon}_i^{(2)}$	$i=i'$	$i\leftrightarrow i'$
a/bAA	$-\eta_{(2)}\,\varepsilon_{i'}^{(2)}$	$\eta_{(2)}\left(\tilde{\varepsilon}_{i'}^{(2)}-\varepsilon_{i'}^{(2)}\right)$	$1,4$	$2+2b,3-3b$
a/bAB	$\eta_{(2)}\,\tilde{\varepsilon}_{i'}^{(2)}$	$\eta_{(2)}\,\varepsilon_{i'}^{(2)}$		
a/bBB	$\eta_{(2)}\left(\tilde{\varepsilon}_{i'}^{(2)}-\varepsilon_{i'}^{(2)}\right)$	$\eta_{(2)}\,\tilde{\varepsilon}_{i'}^{(2)}$		

Table 22: Orientifold projection on exceptional cycles in the $\mathbb{Z}_2^{(2)}$ twisted sector on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ with discrete torsion.

$\Omega\mathcal{R}$ **on exceptional 3-cycles for** $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$**, Part III**				
3-cycle	$\varepsilon_i^{(3)}$	$\tilde{\varepsilon}_i^{(3)}$	$i=i'$	$i\leftrightarrow i'$
a/bAA	$-\eta_{(3)}\,\varepsilon_{i'}^{(3)}$	$\eta_{(3)}\left(\tilde{\varepsilon}_{i'}^{(3)}-\varepsilon_{i'}^{(3)}\right)$	$1,4$	$2+2b,3-3b$
a/bAB	$-\eta_{(3)}\,\tilde{\varepsilon}_{i'}^{(3)}$	$-\eta_{(3)}\,\varepsilon_{i'}^{(3)}$		
a/bBB	$\eta_{(3)}\left(\tilde{\varepsilon}_{i'}^{(3)}-\varepsilon_{i'}^{(3)}\right)$	$\eta_{(3)}\,\tilde{\varepsilon}_{i'}^{(3)}$		

Table 23: Orientifold projection on exceptional cycles in the $\mathbb{Z}_2^{(3)}$ twisted sector on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ with discrete torsion.

The coefficients can be classified as follows: each coefficient can receive contributions from one (I) or two (II) fixed points, and

- for the twist sector $\alpha = 1$, exactly three pairs $(x_i^{(1)}, y_i^{(1)})$ are non-vanishing for $(\sigma_2,\sigma_3)\neq(0,0)$, and four pairs for $(\sigma_2,\sigma_3)=(0,0)$. The remaining three (or two) pairs have zero entries.

$(x_i^{(1)}, y_i^{(1)})$	
I.	II.
$\pm(n^1,m^1)$	$\pm z\,(n^1,m^1)$

with $$z=\left\{\begin{array}{c} 1+(-1)^{\tau_2} \\ 1+(-1)^{\tau_3} \\ (-1)^{\tau_2}+(-1)^{\tau_3} \end{array}\right\}\in\{0,\pm 2\}.$$

The global signs depend on the choice of the $\mathbb{Z}_2^{(1)}$ eigenvalue and the discrete Wilson lines (τ_2,τ_3) as detailed in table 52.

- per twist sector $\alpha \in \{2,3\}$, exactly two of four pairs $(x_i^{(\alpha)}, y_i^{(\alpha)})$ are non-vanishing. The remaining two pairs with $i \in \{1 \ldots 4\}$ have zero-entries. As for the $\mathbb{Z}_2^{(1)}$ twisted sector, there are exactly six different possibilities (up to global signs)

$(x_i^{(\alpha)}, y_i^{(\alpha)})$	
I.	II.
$\pm(n^\alpha, m^\alpha)$	$\pm(-z\, n^\alpha + [1-z]\, m^\alpha, [z-1]\, n^\alpha - m^\alpha)$
$\pm(-n^\alpha - m^\alpha, n^\alpha)$	$\pm(n^\alpha + z\, m^\alpha, -z\, n^\alpha + [1-z]\, m^\alpha)$
$\pm(m^\alpha, -n^\alpha - m^\alpha)$	$\pm([1-z]\, n^\alpha - z\, m^\alpha, z\, n^\alpha + m^\alpha)$

  with $z = (-1)^{\tau_\beta}$ and $(\alpha, \beta) \in \{(2,3),(3,2)\}$. The global signs depend on the choice of the $\mathbb{Z}_2^{(\alpha)}$ eigenvalue and the discrete Wilson lines (τ_1, τ_β) as detailed in tables 53 and 54.

The shape of the coefficients $(x_i^{(\alpha)}, y_i^{(\alpha)})$ is relevant for possible simplifications of the K-theory constraint to be discussed below in section 5.3.

In terms of the expansion (93), the twisted RR tadpole cancellation conditions on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ with discrete torsion read on the six different choices of lattice orientations,

a/bAA :

$$\begin{aligned}
&\sum_{i=0}^{3}\left(\sum_a N_a(1-\eta_{(1)})x_{i,a}^{(1)}\right)\varepsilon_i^{(1)} + \sum_{i=0}^{3}\left(\sum_a N_a[(1+\eta_{(1)})y_{i,a}^{(1)} + \eta_{(1)}2b\, x_{i,a}^{(1)}]\right)\tilde{\varepsilon}_i^{(1)} = \\
&-\left(\sum_a N_a(x_{4,a}^{(1)} - \eta_{(1)}x_{5,a}^{(1)})\right)[\varepsilon_4^{(1)} - \eta_{(1)}\varepsilon_5^{(1)} - b\,(\tilde{\varepsilon}_4^{(1)} - \eta_{(1)}\tilde{\varepsilon}_5^{(1)})] \\
&-\left(\sum_a N_a[y_{4,a}^{(1)} + \eta_{(1)}y_{5,a}^{(1)} + b\,(x_{4,a}^{(1)} + \eta_{(1)}x_{5,a}^{(1)})]\right)(\tilde{\varepsilon}_4^{(1)} + \eta_{(1)}\tilde{\varepsilon}_5^{(1)}), \\
&\sum_{\alpha=2}^{3}\sum_{i=1,4}\left\{\left(\sum_a N_a[(1-\eta_{(\alpha)})x_{i,a}^{(\alpha)} - \eta_{(\alpha)}y_{i,a}^{(\alpha)}]\right)\varepsilon_i^{(\alpha)} + \left(\sum_a N_a(1+\eta_{(\alpha)})y_{i,a}^{(\alpha)}\right)\tilde{\varepsilon}_i^{(\alpha)}\right\} = \\
&-\sum_{\alpha=2}^{3}\left(\sum_a N_a[x_{2+2b,a}^{(\alpha)} - \eta_{(\alpha)}x_{3-2b,a}^{(\alpha)} + \frac{1}{2}(y_{2+2b,a}^{(\alpha)} - \eta_{(\alpha)}y_{3-2b,a}^{(\alpha)})]\right)(\varepsilon_{2+2b}^{(\alpha)} - \eta_{(\alpha)}\varepsilon_{3-2b}^{(\alpha)}) \\
&-\sum_{\alpha=2}^{3}\left(\sum_a N_a(y_{2+2b,a}^{(\alpha)} + \eta_{(2)}y_{3-2b,a}^{(\alpha)})\right)[\tilde{\varepsilon}_{2+2b}^{(\alpha)} - \eta_{(\alpha)}\tilde{\varepsilon}_{3-2b}^{(\alpha)} - \frac{1}{2}(\varepsilon_{2+2b}^{(\alpha)} + \eta_{(\alpha)}\varepsilon_{3-2b}^{(\alpha)})],
\end{aligned} \tag{94}$$

$$
\begin{aligned}
&\mathbf{a/bAB}: \\
&\sum_{i=0,1,2,5}\left(\sum_a N_a(1-\eta_{(1)})x^{(1)}_{i,a}\right)\varepsilon^{(1)}_i+\sum_{i=0,1,2,5}\left(\sum_a N_a[(1+\eta_{(1)})y^{(1)}_{i,a}+\eta_{(1)}2b\,x^{(1)}_{i,a}]\right)\tilde{\varepsilon}^{(1)}_i= \\
&-\left(\sum_a N_a(x^{(1)}_{3,a}-\eta_{(1)}x^{(1)}_{4,a})\right)[\varepsilon^{(1)}_3-\eta_{(1)}\varepsilon^{(1)}_4-b\,(\tilde{\varepsilon}^{(1)}_3-\eta_{(1)}\tilde{\varepsilon}^{(1)}_4)] \\
&-\left(\sum_a N_a[y^{(1)}_{3,a}+\eta_{(1)}y^{(1)}_{4,a}+b\,(x^{(1)}_{3,a}+\eta_{(1)}x^{(1)}_{4,a})]\right)(\tilde{\varepsilon}^{(1)}_3+\eta_{(1)}\tilde{\varepsilon}^{(1)}_4), \\
&\sum_{\alpha=2}^3\sum_{i=1,4}\left(\sum_a N_a(x^{(\alpha)}_{i,a}+(-1)^\alpha\,\eta_{(\alpha)}\,y^{(\alpha)}_{i,a})\right)[\varepsilon^{(\alpha)}_i+(-1)^\alpha\,\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_i]= \\
&-\sum_{\alpha=2}^3\left(\sum_a N_a(x^{(\alpha)}_{2+2b,a}+(-1)^\alpha\,\eta_{(\alpha)}y^{(\alpha)}_{3-2b,a})\right)(\varepsilon^{(\alpha)}_{2+2b}+(-1)^\alpha\,\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_{3-2b}) \\
&-\sum_{\alpha=2}^3\left(\sum_a N_a(x^{(\alpha)}_{3-2b,a}+(-1)^\alpha\,\eta_{(\alpha)}y^{(\alpha)}_{2+2b,a})\right)(\varepsilon^{(\alpha)}_{3-2b}+(-1)^\alpha\,\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_{2+2b}),
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
&\mathbf{a/bBB}: \\
&\sum_{i=0,1,2,4}\left(\sum_a N_a(1-\eta_{(1)})x^{(1)}_{i,a}\right)\varepsilon^{(1)}_i+\sum_{i=0,1,2,4}\left(\sum_a N_a[(1+\eta_{(1)})y^{(1)}_{i,a}+\eta_{(1)}2b\,x^{(1)}_{i,a}]\right)\tilde{\varepsilon}^{(1)}_i= \\
&-\left(\sum_a N_a(x^{(1)}_{3,a}-\eta_{(1)}x^{(1)}_{5,a})\right)[\varepsilon^{(1)}_3-\eta_{(1)}\varepsilon^{(1)}_5-b\,(\tilde{\varepsilon}^{(1)}_3-\eta_{(1)}\tilde{\varepsilon}^{(1)}_5)] \\
&-\left(\sum_a N_a[y^{(1)}_{3,a}+\eta_{(1)}y^{(1)}_{5,a}+b\,(x^{(1)}_{3,a}+\eta_{(1)}x^{(1)}_{5,a})]\right)(\tilde{\varepsilon}^{(1)}_3+\eta_{(1)}\tilde{\varepsilon}^{(1)}_5), \\
&\sum_{\alpha=2}^3\sum_{i=1,4}\left\{\left(\sum_a N_a(1-\eta_{(\alpha)})x^{(\alpha)}_{i,a}\right)\varepsilon^{(\alpha)}_i+\left(\sum_a N_a[(1+\eta_{(\alpha)})y^{(\alpha)}_{i,a}+\eta_{(\alpha)}x^{(\alpha)}_{i,a}]\right)\tilde{\varepsilon}^{(\alpha)}_i\right\}= \\
&-\sum_{\alpha=2}^3\left(\sum_a N_a(x^{(\alpha)}_{2+2b,a}-\eta_{(\alpha)}x^{(\alpha)}_{3-2b,a})\right)[\varepsilon^{(\alpha)}_{2+2b}-\eta_{(\alpha)}\varepsilon^{(\alpha)}_{3-2b}-\frac{1}{2}(\tilde{\varepsilon}^{(\alpha)}_{2+2b}-\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_{3-2b})] \\
&-\sum_{\alpha=2}^3\left(\sum_a N_a[y^{(\alpha)}_{2+2b,a}+\eta_{(\alpha)}y^{(\alpha)}_{3-2b,a}+\frac{1}{2}(x^{(\alpha)}_{2+2b,a}+\eta_{(\alpha)}x^{(\alpha)}_{3-2b,a})]\right)(\tilde{\varepsilon}^{(\alpha)}_{2+2b}+\eta_{(\eta)}\tilde{\varepsilon}^{(\alpha)}_{3-2b}).
\end{aligned}
\tag{96}
$$

We will use the bulk and twisted RR tadpole cancellation conditions (88) and (94) to (96) in section 5.6 to construct an example of a globally consistent D6-brane model on the $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ orientifold with discrete torsion.

5.3 The K-theory constraint

The discussion of the K-theory constraint follows closely the one for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega\mathcal{R})$ in section 3.3. The classification of $\Omega\mathcal{R}$ invariant three-cycles turns out to be given again by table 10, and the exceptional building blocks of the three-cycles in (52) are listed in tables 24 to 26.

Exceptional contributions to $\Omega\mathcal{R}$ invariant 3-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ on a/bAA	
a/bAA : $\mathbb{Z}_2^{(1)}$	
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,0)}$	$\frac{1}{1-b}\left\{[\varepsilon_0^{(1)} - b\tilde{\varepsilon}_0^{(1)}] + (-1)^{\tau_2}[\varepsilon_1^{(1)} - b\tilde{\varepsilon}_1^{(1)}] + (-1)^{\tau_3}[\varepsilon_2^{(1)} - b\tilde{\varepsilon}_2^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_3^{(1)} - b\tilde{\varepsilon}_3^{(1)}]\right\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,0)}$	$\frac{1}{1-b}\left\{[1+(-1)^{\tau_2}][\varepsilon_1^{(1)} - b\tilde{\varepsilon}_1^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_3}[\varepsilon_5^{(1)} - b\tilde{\varepsilon}_5^{(1)}]\right\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,1)}$	$\frac{1}{1-b}\left\{[1+(-1)^{\tau_3}][\varepsilon_2^{(1)} - b\tilde{\varepsilon}_2^{(1)}] + (-1)^{\tau_2}[\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_5^{(1)} - b\tilde{\varepsilon}_5^{(1)}]\right\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,1)}$	$\frac{1}{1-b}\left\{[1+(-1)^{\tau_2+\tau_3}][\varepsilon_3^{(1)} - b\tilde{\varepsilon}_3^{(1)}] + (-1)^{\tau_3}[\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_2}[\varepsilon_5^{(1)} - b\tilde{\varepsilon}_5^{(1)}]\right\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,0)}$	$\tilde{\varepsilon}_0^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_1^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_2^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,0)}$	$[1+(-1)^{\tau_2}]\tilde{\varepsilon}_1^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_4^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,1)}$	$[1+(-1)^{\tau_3}]\tilde{\varepsilon}_2^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_4^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,1)}$	$[1+(-1)^{\tau_2+\tau_3}]\tilde{\varepsilon}_3^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_4^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_5^{(1)}$
a/bAA : $\mathbb{Z}_2^{(i)}$ with $(i,j) \in \{(2,3),(3,2)\}$	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_1,0)}$	$-(-1)^{\tau_j}\left[\varepsilon_{k_1}^{(i)} + (-1)^{\tau_1}\varepsilon_{k_2}^{(i)}\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_1,1)}$	$(-1)^{\tau_j}\varepsilon_{k_1}^{(i)} + [1-(-1)^{\tau_j}]\tilde{\varepsilon}_{k_1}^{(i)} + (-1)^{\tau_1}\left[(-1)^{\tau_j}\varepsilon_{k_2}^{(i)} + [1-(-1)^{\tau_j}]\tilde{\varepsilon}_{k_2}^{(i)}\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_1,0)}$	$(-1)^{\tau_j}\left[-\varepsilon_{k_1}^{(i)} + 2\tilde{\varepsilon}_{k_1}^{(i)} + (-1)^{\tau_1}\left(-\varepsilon_{k_2}^{(i)} + 2\tilde{\varepsilon}_{k_2}^{(i)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_1,1)}$	$[2-(-1)^{\tau_j}]\varepsilon_{k_1}^{(i)} + [-1-(-1)^{\tau_j}]\tilde{\varepsilon}_{k_1}^{(i)} + (-1)^{\tau_1}\left[[2-(-1)^{\tau_j}]\varepsilon_{k_2}^{(i)} + [-1-(-1)^{\tau_j}]\tilde{\varepsilon}_{k_2}^{(i)}\right]$

Table 24: Exceptional three-cycles which enter the K-theory constraint on the **a/bAA** lattice for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ with discrete torsion. All cycles are multiplied by the $\mathbb{Z}_2^{(i)}$ eigenvalue $(-1)^{\tau_0^{(i)}}$. The subscript for the $\mathbb{Z}_2^{(i)}$ sector labels if the 1-cycle is parallel to the $\Omega\mathcal{R}$-plane (h) or perpendicular to it (v) and lists the discrete displacements (σ_j, σ_k) on $T_j \times T_k$.

The bulk parts of the cycles are, as for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \Omega\mathcal{R})$, parallel to the O6-planes with bulk wrapping numbers given in table 19.

Exceptional contributions to $\Omega\mathcal{R}$ invariant 3-cycles for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ on a/bAB	
$\mathbf{a/bAB}: \mathbb{Z}_2^{(1)}$	
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,0)}$	$\frac{1}{1-b}\Big\{[\varepsilon_0^{(1)} - b\tilde{\varepsilon}_0^{(1)}] + (-1)^{\tau_2}[\varepsilon_1^{(1)} - b\tilde{\varepsilon}_1^{(1)}] + (-1)^{\tau_3}[\varepsilon_2^{(1)} - b\tilde{\varepsilon}_2^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_5^{(1)} - b\tilde{\varepsilon}_5^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,0)}$	$\frac{1}{1-b}\Big\{[1+(-1)^{\tau_2}][\varepsilon_1^{(1)} - b\tilde{\varepsilon}_1^{(1)}] + (-1)^{\tau_3}[\varepsilon_3^{(1)} - b\tilde{\varepsilon}_3^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}]$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,1)}$	$\frac{1}{1-b}\Big\{[1+(-1)^{\tau_3}][\varepsilon_2^{(1)} - b\tilde{\varepsilon}_2^{(1)}] + (-1)^{\tau_2}[\varepsilon_3^{(1)} - b\tilde{\varepsilon}_3^{(1)}] + (-1)^{\tau_2+\tau_3}[\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,1)}$	$\frac{1}{1-b}\Big\{(-1)^{\tau_2+\tau_3}[\varepsilon_3^{(1)} - b\tilde{\varepsilon}_3^{(1)}] + [\varepsilon_4^{(1)} - b\tilde{\varepsilon}_4^{(1)}] + [(-1)^{\tau_2}+(-1)^{\tau_3}][\varepsilon_5^{(1)} - b\tilde{\varepsilon}_5^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,0)}$	$\tilde{\varepsilon}_0^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_1^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_2^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,0)}$	$[1+(-1)^{\tau_2}]\tilde{\varepsilon}_1^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_3^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_4^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,1)}$	$[1+(-1)^{\tau_3}]\tilde{\varepsilon}_2^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_3^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_4^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,1)}$	$(-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)} + \tilde{\varepsilon}_4^{(1)} + [(-1)^{\tau_2}+(-1)^{\tau_3}]\tilde{\varepsilon}_5^{(1)}$
$\mathbf{a/bAB}: \mathbb{Z}_2^{(2)}$	
$\Pi^{\mathbb{Z}_2^{(2)}}_{h,(\sigma_1,0)}$	$(-1)^{\tau_3}\left[-\varepsilon_{k_1}^{(2)} + \tilde{\varepsilon}_{k_1}^{(2)} + (-1)^{\tau_1}\left(-\varepsilon_{k_2}^{(2)} + \tilde{\varepsilon}_{k_2}^{(2)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(2)}}_{h,(\sigma_1,1)}$	$\varepsilon_{k_1}^{(2)} - (-1)^{\tau_3}\tilde{\varepsilon}_{k_1}^{(2)} + (-1)^{\tau_1}\left[\varepsilon_{k_2}^{(2)} - (-1)^{\tau_3}\tilde{\varepsilon}_{k_2}^{(2)}\right]$
$\Pi^{\mathbb{Z}_2^{(2)}}_{v,(\sigma_1,0)}$	$(-1)^{\tau_3}\left[\varepsilon_{k_1}^{(2)} + \tilde{\varepsilon}_{k_1}^{(2)} + (-1)^{\tau_1}\left(\varepsilon_{k_2}^{(2)} + \tilde{\varepsilon}_{k_2}^{(2)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(2)}}_{v,(\sigma_1,1)}$	$[1-2(-1)^{\tau_3}]\varepsilon_{k_1}^{(2)} + [-2+(-1)^{\tau_3}]\tilde{\varepsilon}_{k_1}^{(2)} + (-1)^{\tau_1}\left[[1-2(-1)^{\tau_3}]\varepsilon_{k_2}^{(2)} + [-2+(-1)^{\tau_3}]\tilde{\varepsilon}_{k_2}^{(2)}\right]$
$\mathbf{a/bAB}: \mathbb{Z}_2^{(3)}$	
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(\sigma_1,0)}$	$(-1)^{\tau_2}\left[\varepsilon_{k_1}^{(3)} + \tilde{\varepsilon}_{k_1}^{(3)} + (-1)^{\tau_1}\left(\varepsilon_{k_2}^{(3)} + \tilde{\varepsilon}_{k_2}^{(3)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(\sigma_1,1)}$	$[1-2(-1)^{\tau_2}]\varepsilon_{k_1}^{(3)} + [-2+(-1)^{\tau_2}]\tilde{\varepsilon}_{k_1}^{(3)} + (-1)^{\tau_1}\left[[1-2(-1)^{\tau_2}]\varepsilon_{k_2}^{(3)} + [-2+(-1)^{\tau_2}]\tilde{\varepsilon}_{k_2}^{(3)}\right]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(\sigma_1,0)}$	$(-1)^{\tau_2}\left[\varepsilon_{k_1}^{(3)} - \tilde{\varepsilon}_{k_1}^{(3)} + (-1)^{\tau_1}\left(\varepsilon_{k_2}^{(3)} - \tilde{\varepsilon}_{k_2}^{(3)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(\sigma_1,1)}$	$-\varepsilon_{k_1}^{(3)} + (-1)^{\tau_2}\tilde{\varepsilon}_{k_1}^{(3)} + (-1)^{\tau_1}\left[-\varepsilon_{k_2}^{(3)} + (-1)^{\tau_2}\tilde{\varepsilon}_{k_2}^{(3)}\right]$

Table 25: Exceptional three-cycles which enter the K-theory constraint on the **a/bAB** lattice for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ with discrete torsion. Observe that the calculations can be simplified by noting that $(\Pi^{\mathbb{Z}_2^{(2)}}_{h,(\sigma_1,\sigma_3)}, \Pi^{\mathbb{Z}_2^{(2)}}_{v,(\sigma_1,\sigma_3)}) \Leftrightarrow (-\Pi^{\mathbb{Z}_2^{(3)}}_{v,(\sigma_1,\sigma_2)}, \Pi^{\mathbb{Z}_2^{(3)}}_{h,(\sigma_1,\sigma_2)})$ when the same values for σ_2 and σ_3 are taken. For details on the notation see table 24.

In terms of the expansion (93) of a fractional three-cycle on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$, the intersection

Exceptional contributions to $\Omega\mathcal{R}$ invariant 3-cycles for $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$ on a/bBB	
a/bBB : $\mathbb{Z}_2^{(1)}$	
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,0)}$	$\frac{1}{1-b}\Big\{[\varepsilon_0^{(1)}-b\tilde{\varepsilon}_0^{(1)}]+(-1)^{\tau_2}[\varepsilon_1^{(1)}-b\tilde{\varepsilon}_1^{(1)}]+(-1)^{\tau_3}[\varepsilon_2^{(1)}-b\tilde{\varepsilon}_2^{(1)}]+(-1)^{\tau_2+\tau_3}[\varepsilon_4^{(1)}-b\tilde{\varepsilon}_4^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,0)}$	$\frac{1}{1-b}\Big\{[1+(-1)^{\tau_2}][\varepsilon_1^{(1)}-b\tilde{\varepsilon}_1^{(1)}]+(-1)^{\tau_2+\tau_3}[\varepsilon_3^{(1)}-b\tilde{\varepsilon}_3^{(1)}]+(-1)^{\tau_3}[\varepsilon_5^{(1)}-b\tilde{\varepsilon}_5^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,1)}$	$\frac{1}{1-b}\Big\{[1+(-1)^{\tau_3}][\varepsilon_2^{(1)}-b\tilde{\varepsilon}_2^{(1)}]+(-1)^{\tau_2+\tau_3}[\varepsilon_3^{(1)}-b\tilde{\varepsilon}_3^{(1)}]+(-1)^{\tau_2}[\varepsilon_5^{(1)}-b\tilde{\varepsilon}_5^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,1)}$	$\frac{1}{1-b}\Big\{[\varepsilon_3^{(1)}-b\tilde{\varepsilon}_3^{(1)}]+[(-1)^{\tau_2}+(-1)^{\tau_3}][\varepsilon_4^{(1)}-b\tilde{\varepsilon}_4^{(1)}]+(-1)^{\tau_2+\tau_3}[\varepsilon_5^{(1)}-b\tilde{\varepsilon}_5^{(1)}]\Big\}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,0)}$	$\tilde{\varepsilon}_0^{(1)}+(-1)^{\tau_2}\tilde{\varepsilon}_1^{(1)}+(-1)^{\tau_3}\tilde{\varepsilon}_2^{(1)}+(-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_4^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,0)}$	$[1+(-1)^{\tau_2}]\tilde{\varepsilon}_1^{(1)}+(-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)}+(-1)^{\tau_3}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,1)}$	$[1+(-1)^{\tau_3}]\tilde{\varepsilon}_2^{(1)}+(-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)}+(-1)^{\tau_2}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,1)}$	$\tilde{\varepsilon}_3^{(1)}+[(-1)^{\tau_2}+(-1)^{\tau_3}]\tilde{\varepsilon}_4^{(1)}+(-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_5^{(1)}$
a/bBB : $\mathbb{Z}_2^{(i)}$ with $(i,j)\in\{(2,3),(3,2)\}$	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_1,0)}$	$(-1)^{\tau_j}\left[-2\varepsilon_{k_1}^{(i)}+\tilde{\varepsilon}_{k_1}^{(i)}+(-1)^{\tau_1}\left(-2\varepsilon_{k_2}^{(i)}+\tilde{\varepsilon}_{k_2}^{(i)}\right)\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_1,1)}$	$[1+(-1)^{\tau_j}]\varepsilon_{k_1}^{(i)}+[1-2(-1)^{\tau_j}]\tilde{\varepsilon}_{k_1}^{(i)}+(-1)^{\tau_1}\left[[1+(-1)^{\tau_j}]\varepsilon_{k_2}^{(i)}+[1-2(-1)^{\tau_j}]\tilde{\varepsilon}_{k_2}^{(i)}\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_1,0)}$	$(-1)^{\tau_j}\left[\tilde{\varepsilon}_{k_1}^{(i)}+(-1)^{\tau_1}\tilde{\varepsilon}_{k_2}^{(i)}\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_1,1)}$	$[1-(-1)^{\tau_j}]\varepsilon_{k_1}^{(i)}-\tilde{\varepsilon}_{k_1}^{(i)}+(-1)^{\tau_1}\left[[1-(-1)^{\tau_j}]\varepsilon_{k_2}^{(i)}-\tilde{\varepsilon}_{k_2}^{(i)}\right]$

Table 26: Exceptional three-cycles which enter the K-theory constraint on the **a/bBB** lattice for $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ with discrete torsion. For details on the notation see table 24.

numbers read

$$\begin{aligned}
\Pi_a^{\rm frac}\circ\Pi_b^{\rm frac} =&\frac{1}{4}\left(2\left[P_aU_b-P_bU_a+Q_aV_b-Q_bV_a\right]+P_aV_b-P_bV_a+Q_aU_b-Q_bU_a\right)\\
&-\frac{1}{4}\sum_{\alpha=1}^{3}\left(\vec{x}_a^{(\alpha)}\cdot\vec{y}_b^{(\alpha)}-\vec{x}_b^{(\alpha)}\cdot\vec{y}_a^{(\alpha)}\right)\\
&\text{with}\quad \vec{x}_a^{(1)}\cdot\vec{y}_b^{(1)}\equiv 3\,x_{0,a}^{(1)}y_{0,b}^{(1)}+\sum_{m=1}^{5}x_{m,a}^{(1)}y_{m,b}^{(1)}\\
&\text{and}\quad \vec{x}_a^{(\alpha)}\cdot\vec{y}_b^{(\alpha)}\equiv\sum_{m=1}^{4}x_{m,a}^{(\alpha)}y_{m,b}^{(\alpha)}\qquad \alpha\in\{2,3\}.
\end{aligned}\tag{97}$$

The bulk parts of the intersection numbers, both without and with discrete torsion, are

explicitly given in table 27. They can be simplified by imposing the bulk RR tadpole

Bulk parts of K-theory constraints on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$				
	lattice	**a/bAA**	**a/bAB**	**a/bBB**
$\Omega\mathcal{R}$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_0}^{\text{bulk}}}{2^{3-\eta}}$	$\frac{-\sum_a N_a(2U_a+V_a+b(2P_a+Q_a))}{2^{1-\eta}(1-b)}$	$\frac{-3\sum_a N_a(U_a+V_a+b(P_a+Q_a))}{2^{1-\eta}(1-b)}$	$\frac{-3\sum_a N_a(U_a+2V_a+b(P_a+2Q_a))}{2^{1-\eta}(1-b)}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_1}^{\text{bulk}}}{2^{3-\eta}}$	$\frac{-3\sum_a N_a(2U_a+V_a+b(2P_a+Q_a))}{2^{1-\eta}(1-b)}$	$\frac{-3\sum_a N_a(U_a+V_a+b(P_a+Q_a))}{2^{1-\eta}(1-b)}$	$\frac{-\sum_a N_a(U_a+2V_a+b(P_a+2Q_a))}{2^{1-\eta}(1-b)}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_2}^{\text{bulk}}}{2^{3-\eta}}$	$-\frac{3}{2^{1-\eta}} \sum_a N_a Q_a$	$\frac{3}{2^{1-\eta}} \sum_a N_a (P_a - Q_a)$	$\frac{3\sum_a N_a P_a}{2^{1-\eta}}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$\frac{\sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_3}^{\text{bulk}}}{2^{3-\eta}}$	$-\frac{3}{2^{1-\eta}} \sum_a N_a Q_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (P_a - Q_a)$	$\frac{3\sum_a N_a P_a}{2^{1-\eta}}$

Table 27: Bulk part of the K-theory constraints on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ without ($\eta = 1$) and with discrete torsion ($\eta = -1$). By inserting the bulk RR tadpole cancellation conditions (88), the simpler form in table 28 appears.

cancellation conditions (88). This leads to the expressions in table 28. On can immediately

Simplification of the bulk K-theory constraints on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$					
a/bAA	after RR tcc mod 2	**a/bAB**	after RR tcc mod 2	**a/bBB**	after RR tcc mod 2
$\Omega\mathcal{R}$	$\frac{-\sum_a N_a(2U_a+V_a)}{2^{1-\eta}(1-b)}$	$\Omega\mathcal{R}$	$\frac{-3\sum_a N_a(U_a+V_a)}{2^{1-\eta}(1-b)}$	$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$\frac{-\sum_a N_a(U_a+2V_a)}{2^{1-\eta}(1-b)}$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$3 \cdot 2^\eta \sum_a N_a P_a$	$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$2^\eta \sum_a N_a P_a$	$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$-3 \cdot 2^\eta \sum_a N_a Q_a$

Table 28: Simplification of the K-theory constraints on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ upon the bulk RR tadpole cancellation conditions (88). The remaining two kinds of probe branes are proportional to the constraints in this table, where the integer multiplicities can be read off from table 27.

see that in the case without torsion ($\eta = 1$) and a tilted two-torus $T^2_{(1)}$, the K-theory constraints are trivially fulfilled for all models that satisfy RR tadpole cancellation. For trivial torsion ($\eta = 1$) and an untilted $T^2_{(1)}$, still half of the K-theory constraints are trivial. In the case with discrete torsion ($\eta = -1$), the exceptional contributions of the intersection numbers according to equation (97) have to be added, which can again be simplified using the twisted RR tadpole cancellation conditions (94) to (96) and the knowledge of the shape of the coefficients $(x_i^{(\alpha)}, y_i^{(\alpha)})$ as discussed in section 5.2. Due to the large number of

combinations of the exceptional cycles in tables 24 to 26 with $\Omega\mathcal{R}$ and $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ invariant bulk three-cycles, we do not write out explicitly the K-theory constraints on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ here.

Instead, we give examples of consistent models where the K-theory constraints are fulfilled trivially by only having even ranks of the gauge factors.

5.4 Exceptional three-cycles at $\mathbb{Z}_3$ singularities

The discrete torsion factor acts trivially in the $\mathbb{Z}_3$ twisted sector, as displayed in table 3. The exceptional three-cycles in this sector, for both choices of torsion, are given by

$$\begin{aligned} \delta^{(k)} &= (d_{22}+d_{33}-d_{23}-d_{32})\otimes\pi_1,\\ \tilde{\delta}^{(k)} &= (d_{22}+d_{33}-d_{23}-d_{32})\otimes\pi_2,\\ \text{with}\quad \delta^{(k)}\circ\tilde{\delta}^{(l)} &= \left(\begin{array}{cc} -2 & 1\\ 1 & -2\end{array}\right)_{kl}, \end{aligned} \tag{98}$$

and all others vanishing. These exceptional three-cycles are relevant for determining an unimodular basis of the lattice of three-cycles, but cannot be used for model building with D6-branes as mentioned in section 2. We will therefore not discuss them further in this article.

5.5 A $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$ example without torsion

In [68, 69, 79], two classes of supersymmetric models on **aAB** on $T^6/\mathbb{Z}_6'$ were analysed with complex structure parameters $\varrho\equiv\sqrt{3}\frac{R_2}{R_1}=\frac{1}{\omega}$ and $\omega=1$ for the models with hidden sectors and $\omega=2$ for the model without hidden sector. The representants per orbifold and orientifold orbit, as discussed at the end of section 5.1, are listed in table 29. On the **aAB** lattice without discrete torsion, the untwisted RR tadpole cancellation conditions (88) read

$$\begin{aligned} \sum_a N_a\,(P_a+Q_a) &= 8,\\ \sum_a N_a\,(U_a-V_a) &= 16, \end{aligned} \tag{100}$$

which can be solved by

$$\Pi^{\text{frac}}_{\tilde{b}} = \frac{1}{2}\,(\rho_1+2\,\rho_3) \qquad \text{with} \qquad N_{\tilde{b}}=8 \tag{101}$$

Supersymmetric bulk 3-cycles on aAB on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$						
brane/orbit	$\frac{\text{angle}}{\pi}$	$(n^1, m^1; n^2, m^2; n^3, m^3)$	P	Q	U	V
$a, h_i/\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(0, 1; 1, 0; 1, -1)$	0	0	1	-1
b	$(\frac{1}{6}, -\frac{1}{6}, 0)$	$(1, \omega; 2, -1; 1, 1)$	3	0	$3\,\omega$	0
$c/\Omega\mathcal{R}$	$(0, 0, 0)$	$(1, 0; 1, 0; 1, 1)$	1	1	0	0
$d, \hat{h}_1/\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(0, 1; 1, -2; 1, 1)$	0	0	3	-3
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$(0, \frac{1}{2}, -\frac{1}{2})$	$(1, 0; -1, 2; 1, -1)$	1	1	0	0
$\tilde{b}$	$(\frac{1}{6}, \frac{1}{3}, -\frac{1}{2})$	$(1, \omega; 0, 1; 1, -1)$	1	0	ω	0

(99)

Table 29: Examples of supersymmetric bulk three-cycles on the **aAB** lattice on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6\times\Omega\mathcal{R})$. The labels $a, \ldots, d, h_i$ and $\hat{h}_1$ correspond to the cycles used for model building on $T^6/\mathbb{Z}_6'$ in [79] up to permutation of tori. $\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$ denotes the fact that cycle a is parallel to this O6-plane orbit. The complex structures, for which b and $\tilde{b}$ are supersymmetric, are $\varrho = \frac{1}{\omega}$.

for $\omega = 2$ and the complex structure $\varrho = \frac{1}{2}$. The resulting gauge group is $U(8)$ with the Abelian subgroup $U(1) \subset U(8)$ anomalous and massive, and the chiral spectrum is derived from the intersection numbers

$$\begin{aligned} \Pi_{\tilde{b}}^{\text{frac}} \circ \Pi_{\tilde{b}'}^{\text{frac}} &= -4, \\ \Pi_{\tilde{b}}^{\text{frac}} \circ \Pi_{O6} &= -8, \end{aligned} \tag{102}$$

as

$$6\,(\overline{\mathbf{28}}) + 2\,(\mathbf{36}). \tag{103}$$

The non-chiral massless spectrum is computed along the lines describe in appendix B and gives

- three multiplets in the adjoint representation ($\mathbf{64}$) of $U(8)$ from the $\tilde{b}\tilde{b}$ sector and two further multiplets in the adjoint at $\tilde{b}(\omega^k\,\tilde{b})_{k=1,2}$ intersections,
- six non-chiral pairs of antisymmetrics $6 \times [(\mathbf{28}) + c.c.]$ of $U(8)$ at $\tilde{b}(\omega^k\tilde{b}')_{k=0,1,2}$ intersections.

The K-theory constraint is trivially fulfilled for a single gauge factor $U(8)$ due to its even rank.

5.6 A $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6 \times \Omega\mathcal{R})$ example with discrete torsion and D6-branes

On the **aAB** lattice with discrete torsion and the choice $\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(2)}} = -1$ of the exotic O6-plane, the untwisted RR tadpole cancellation conditions (88) read

$$\begin{aligned} \sum_a N_a \left(P_a + Q_a\right) = 16, \\ \sum_a N_a \left(U_a - V_a\right) = 16, \end{aligned} \tag{104}$$

which can be solved by four kinds of fractional D6-branes adding up to the supersymmetric bulk cycle $\tilde{b}$ with $\omega = 1$ in table 29 with complex structure $\varrho = 1$ and $N_{\tilde{b}_m} = 4$ for $m = 0 \dots 3$,

$$\begin{aligned} \Pi^{\text{frac}}_{\tilde{b}_m} =& \frac{1}{4}\left(\rho_1 + \rho_3\right) + \frac{(-1)^{\tau^{(1)}_{0,m}}}{4}\left(2\,\varepsilon^{(1)}_1 + 2\,\tilde{\varepsilon}^{(1)}_1 + \varepsilon^{(1)}_4 + \tilde{\varepsilon}^{(1)}_4 + \varepsilon^{(1)}_5 + \tilde{\varepsilon}^{(1)}_5\right) \\ &+ \frac{(-1)^{\tau^{(2)}_{0,m}}}{4}\left(-\varepsilon^{(2)}_1 - \varepsilon^{(2)}_3\right) + \frac{(-1)^{\tau^{(3)}_{0,m}}}{4}\left(\varepsilon^{(3)}_1 + \varepsilon^{(3)}_3\right), \end{aligned} \tag{105}$$

where we choose the discrete displacement $\vec{\sigma} = (0,1,0)$ and no Wilson line, $\vec{\tau} = \vec{0}$, for all four D6-branes.

The orientifold image branes are

$$\begin{aligned} \Pi^{\text{frac}}_{\tilde{b}'_m} =& \frac{1}{4}\left(\rho_2 - \rho_4\right) + \frac{(-1)^{\tau^{(1)}_{0,m}}}{4}\left(-2\,\varepsilon^{(1)}_1 + 2\,\tilde{\varepsilon}^{(1)}_1 - \varepsilon^{(1)}_3 + \tilde{\varepsilon}^{(1)}_3 - \varepsilon^{(1)}_5 + \tilde{\varepsilon}^{(1)}_5\right) \\ &- \frac{(-1)^{\tau^{(2)}_{0,m}}}{4}\left(-\tilde{\varepsilon}^{(2)}_1 - \tilde{\varepsilon}^{(2)}_3\right) + \frac{(-1)^{\tau^{(3)}_{0,m}}}{4}\left(-\tilde{\varepsilon}^{(3)}_1 - \tilde{\varepsilon}^{(3)}_3\right), \end{aligned} \tag{106}$$

where we used $\eta_{(2)} = -1$ and $\eta_{(1)} = \eta_{(3)} = 1$.

The relevant intersection numbers are thus

$$\begin{aligned} \Pi^{\text{frac}}_{\tilde{b}_m} \circ \Pi^{\text{frac}}_{\tilde{b}_n} =& 0, \\ \Pi^{\text{frac}}_{\tilde{b}_m} \circ \Pi^{\text{frac}}_{\tilde{b}'_n} =& -\frac{1}{2} - \frac{5}{2}(-1)^{\tau^{(1)}_{0,m}+\tau^{(1)}_{0,n}} + \frac{1}{2}(-1)^{\tau^{(2)}_{0,m}+\tau^{(2)}_{0,n}} + \frac{1}{2}(-1)^{\tau^{(3)}_{0,m}+\tau^{(3)}_{0,n}} \\ =& \left\{ \begin{array}{cc} -2 & (m,m) \\ -4 & (0,1),(2,3) \\ 2 & (0,2),(1,3),(0,3),(1,2) \end{array} \right., \\ \Pi^{\text{frac}}_{\tilde{b}_m} \circ \Pi_{O6} =& -2, \end{aligned} \tag{107}$$

where we choose the following assignment of $\mathbb{Z}_2^{(i)}$ eigenvalues.

brane	$\tau_{0,m}^{(1)}$	$\tau_{0,m}^{(2)}$	$\tau_{0,m}^{(3)}$
$\tilde{b}_0$	0	0	0
$\tilde{b}_1$	0	1	1
$\tilde{b}_2$	1	0	1
$\tilde{b}_3$	1	1	0

This means that there is no net-chirality in the $(\mathbf{N}_m, \overline{\mathbf{N}}_n)_{m\neq n}$ sectors, but in the $(\mathbf{N}_m, \mathbf{N}_n)_{m\neq n}$ sectors, as well as in $(\mathbf{Anti}_m)$. The chiral spectrum of $U(4)^4$ is in detail

$$\begin{aligned}
&2\,(\overline{\mathbf{6}},\mathbf{1},\mathbf{1},\mathbf{1}) + 2\,(\mathbf{1},\overline{\mathbf{6}},\mathbf{1},\mathbf{1}) + 2\,(,\mathbf{1},\mathbf{1},\overline{\mathbf{6}},\mathbf{1}) + 2\,(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{6}}) \\
&+ 4\,(\overline{\mathbf{4}},\overline{\mathbf{4}},\mathbf{1},\mathbf{1}) + 4\,(,\mathbf{1},\mathbf{1},\overline{\mathbf{4}},\overline{\mathbf{4}}) \\
&+ 2\,(\mathbf{4},\mathbf{1},\mathbf{4},\mathbf{1}) + 2\,(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{4}) + 2\,(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{4}) + 2\,(\mathbf{1},\mathbf{4},\mathbf{4},\mathbf{1}).
\end{aligned} \tag{108}$$

The complete spectrum is computed via the intersection numbers as described in appendix B.2, leading to the addition of the non-chiral matter states

- $\mathcal{N} = 1$ multiplets in bifundamental representations on parallel branes $\tilde{b}_m\tilde{b}_n$,

$$\big[(\mathbf{4},\overline{\mathbf{4}},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{4},\overline{\mathbf{4}}) + (\mathbf{4},\mathbf{1},\overline{\mathbf{4}},\mathbf{1}) + (\mathbf{1},\mathbf{4},\mathbf{1},\overline{\mathbf{4}}) + (\mathbf{4},\mathbf{1},\mathbf{1},\overline{\mathbf{4}}) + (\mathbf{1},\mathbf{4},\overline{\mathbf{4}},\mathbf{1}) + c.c.\big]$$

- more bifundamentals at intersections $\tilde{b}_m(\omega^k\,\tilde{b}_n)_{k=1,2}$ for $m \neq n$,

$$\big[(\mathbf{4},\mathbf{1},\overline{\mathbf{4}},\mathbf{1}) + (\mathbf{1},\mathbf{4},\mathbf{1},\overline{\mathbf{4}}) + (\mathbf{4},\mathbf{1},\mathbf{1},\overline{\mathbf{4}}) + (\mathbf{1},\mathbf{4},\overline{\mathbf{4}},\mathbf{1}) + c.c.\big]$$

- one non-chiral pair of symmetrics per stack of fractional D6-branes,

$$[(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{10}) + c.c.].$$

The $U(1)_m \subset U(4)_m$ gauge factors are all anomalous and acquire a mass by the generalised Green-Schwarz mechanism.

The K-theory constraint is again trivially fulfilled due to the even rank of each gauge group.

Note that there is no adjoint representation in the massless spectrum, which means that the D6-branes in this example are completely rigid.

6 The $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ orientifolds

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ orbifold is generated by the shift vectors

$$\vec{v} = \frac{1}{2}(1,-1,0), \quad \vec{w} = \frac{1}{6}(-2,1,1), \tag{109}$$

and on the compactification lattice $SU(3)^3$, the Hodge numbers relevant for D6-brane model building read

$$\eta = 1: \; h_{21} = 0, \qquad\qquad \eta = -1: \; h_{21} = 15_{\mathbb{Z}_2}. \tag{110}$$

The complete list of Hodge numbers per twist sector is given in table 4, and the decomposition after the orientifold projection is displayed in table 46.

In the case without discrete torsion, there exist only two three-cycles. Since the supersymmetry condition projects onto a one-dimensional sub-space, on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ without discrete torsion, no chiral supersymmetric models can be found. On the other hand, the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ orbifold with discrete torsion contains a $T^6/\mathbb{Z}_6$ sub-sector, and in analogy to the known three-generation models there, we expect to find phenomenologically appealing spectra in the future [87]. In the following, we concentrate on the latter case with discrete torsion.

6.1 The bulk part

The lattice of three-cycles consists solely of bulk and exceptional cycles from the three $\mathbb{Z}_2^{(i)}$ twisted sectors.

The bulk cycles are (up to normalisation) identical to the $T^6/\mathbb{Z}_6$ case of [70, 71]. The two linearly independent cycles are

$$\begin{aligned}
\rho_1 &= 4\left(1+\theta+\theta^2\right)\pi_{135} = 4\left(\pi_{136}+\pi_{145}+\pi_{235}-\pi_{146}-\pi_{245}-\pi_{236}\right), \\
\rho_2 &= 4\left(1+\theta+\theta^2\right)\pi_{136} = 4\left(\pi_{136}+\pi_{145}+\pi_{235}-\pi_{246}-\pi_{135}\right),
\end{aligned} \tag{111}$$

with intersection form

$$\rho_1 \circ \rho_2 = 4. \tag{112}$$

The 1-cycle wrapping numbers transform as

$$\begin{pmatrix} n^1 & m^1 \\ n^2 & m^2 \\ n^3 & m^3 \end{pmatrix} \stackrel{\mathbb{Z}_6'}{\rightarrow} \begin{pmatrix} m^1 & -(n^1+m^1) \\ -m^2 & n^2+m^2 \\ -m^3 & n^3+m^3 \end{pmatrix} \stackrel{\mathbb{Z}_6'}{\rightarrow} \begin{pmatrix} -(n^1+m^1) & n^1 \\ -(n^2+m^2) & n^2 \\ -(n^3+m^3) & n^3 \end{pmatrix} \tag{113}$$

under the $\mathbb{Z}_6'$ symmetry, and the bulk wrapping numbers (X, Y) along ρ_1, ρ_2 are obtained by taking the orbifold invariant orbit,

$$\Pi^{\text{bulk}} = X\ \rho_1 + Y\ \rho_2. \tag{114}$$

This leads to the definition

$$\begin{aligned} X &\equiv n^1 n^2 n^3 - m^1 m^2 m^3 - \sum_{i \neq j \neq k \neq i} n^i m^j m^k, \\ Y &\equiv \sum_{i \neq j \neq k \neq i} \left(n^i n^j m^k + n^i m^j m^k \right). \end{aligned} \tag{115}$$

The bulk supersymmetry conditions read

$$\begin{array}{rll} \textbf{AAA}: & Y_a = 0, & 2X_a + Y_a > 0, \\ \textbf{AAB}: & Y_a - X_a = 0, & X_a + Y_a > 0, \\ \textbf{ABB}: & X_a = 0, & X_a + 2Y_a > 0, \\ \textbf{BBB}: & 2X_a + Y_a = 0, & Y_a > 0, \end{array} \tag{116}$$

on the four inequivalent lattice orientations.

The orientifold projection on the bulk three-cycles is listed in table 30.

$\Omega\mathcal{R}$ **projection on bulk 3-cycles on** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$		
$3 - \text{cycle}$	ρ_1	ρ_2
AAA	ρ_1	$\rho_1 - \rho_2$
AAB	ρ_2	ρ_1
ABB	$\rho_2 - \rho_1$	ρ_2
BBB	$-\rho_1$	$\rho_2 - \rho_1$

Table 30: Orientifold projection on bulk three-cycles on the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ orientifold without and with discrete torsion.

The torus wrapping numbers of the O6-planes are listed in table 31, and the corresponding bulk wrapping numbers are given in table 32.

Torus wrapping numbers of O6-planes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$					
$\mathrm{O-plane}$	$\frac{\text{angle}}{\pi}$	**AAA**	**AAB**	**ABB**	**BBB**
		$(n^1, m^1; n^2, m^2; n^3, m^3)$			
$\Omega\mathcal{R}$	$(0,0,0)$	$(1,0;1,0;1,0)$	$(1,0;1,0;1,1)$	$(1,0;1,1;1,1)$	$(1,1;1,1;1,1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$(0,\frac{1}{2},-\frac{1}{2})$	$(1,0;-1,2;1,-2)$	$(1,0;-1,2;1,-1)$	$(1,0;-1,1;1,-1)$	$(1,1;-1,1;1,-1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2},-\frac{1}{2},0)$	$(-1,2;1,-2;1,0)$	$(-1,2;1,-2;1,1)$	$(-1,2;1,-1;1,1)$	$(-1,1;1,-1;1,1)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$(\frac{1}{2},0,-\frac{1}{2})$	$(-1,2;1,0;1,-2)$	$(-1,2;1,0;1,-1)$	$(-1,2;1,1;1,-1)$	$(-1,1;1,1;1,-1)$

Table 31: Torus wrapping numbers of one representant per O6-plane orbit on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ without and with discrete torsion. In the second column, the angles w.r.t. the $\Omega\mathcal{R}$ invariant axis are given in units of π.

Bulk wrapping numbers of O6-planes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$								
	AAA		**AAB**		**ABB**		**BBB**	
orbit	X	Y	X	Y	X	Y	X	Y
$\Omega\mathcal{R}$	1	0	1	1	0	3	-3	6
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	3	0	1	1	0	1	-1	2
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	3	0	3	3	0	3	-1	2
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	3	0	1	1	0	3	-1	2

Table 32: Bulk wrapping numbers for the four O6-plane orbits on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ without and with discrete torsion computed from the torus wrapping numbers in table 31 using (115).

The bulk RR tadpole cancellation conditions can now be written as

$$\begin{aligned}
\mathbf{AAA}: \quad 0 &= \left[\sum_a N_a(2X_a + Y_a) - 2^{\frac{3-\eta}{2}} \left(\eta_{\Omega\mathcal{R}} + 3\sum_{i=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\right)\right]\rho_1, \\
\mathbf{AAB}: \quad 0 &= \left[\sum_a N_a(X_a + Y_a) - 2^{\frac{3-\eta}{2}} \left(\sum_{i=0}^{2} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} + 3\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}}\right)\right](\rho_1+\rho_2), \\
\mathbf{ABB}: \quad 0 &= \left[\sum_a N_a(X_a + 2Y_a) - 2^{\frac{3-\eta}{2}} \left(3\sum_{i=0,2,3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}\right)\right]\rho_2, \\
\mathbf{BBB}: \quad 0 &= \left[\sum_a N_a Y_a - 2^{\frac{3-\eta}{2}} \left(3\,\eta_{\Omega\mathcal{R}} + \sum_{i=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\right)\right](-\rho_1+2\rho_2),
\end{aligned} \tag{117}$$

with $\eta = \pm 1$ for the case without and with discrete torsion. This is as expected $1 = 1 + h_{21}^U$ condition per lattice.

Double counting of models is on this orbifold background avoided as follows:

- $\boxed{(n^3, m^3) = (\text{odd,odd})}$ selects one $\mathbb{Z}_6'$ orbifold image,
- $\boxed{n^3 > 0}$ fixes the orientation on $T^2_{(3)}$,
- $\boxed{0 \leq m^1 \leq 2|n^1| \text{ on } \mathbf{A} \text{ and } |n^1| \leq m^1 \text{ on } \mathbf{B}}$ fixes the orientation on $T^2_{(1)}$ and singles out a D6-brane compared to its orientifold image,
- if the D6-brane is at $\boxed{\text{angle } 0, \pi/2 \text{ on } T^2_{(1)}}$, the orientifold image can be excluded by demanding that the angle $\pi\phi^{(3)}$ w.r.t. the $\Omega\mathcal{R}$ plane is in the range $(-\frac{\pi}{2}, 0)$; using the relation (32) leads to $\boxed{-2n^3 < m^3 < 0 \text{ on } \mathbf{A} \text{ and } -n^3 < m^3 < n^3 \text{ on } \mathbf{B}}$ on $T^2_{(3)}$,
- D6-branes parallel to some $\Omega\mathcal{R}\theta^n\omega^m$ plane are treated separately; their torus wrapping numbers are given in table 31.

6.2 The $\mathbb{Z}_2$ twisted parts

Each $\mathbb{Z}_2^{(\alpha)}$ twisted sector with $\alpha = 1, 2, 3$ is similar to the $\mathbb{Z}_2$ twisted sector of $T^6/\mathbb{Z}_6$ [70, 71] with a different normalisation factor of the exceptional three-cycles,

$$\begin{aligned}
\varepsilon_1^{(\alpha)} &= 2\left(e_{41}^{(\alpha)} - e_{61}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{61}^{(\alpha)} - e_{51}^{(\alpha)}\right) \otimes \pi_2, & \tilde{\varepsilon}_1^{(\alpha)} &= 2\left(e_{51}^{(\alpha)} - e_{61}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{41}^{(\alpha)} - e_{51}^{(\alpha)}\right) \otimes \pi_2,\\
\varepsilon_2^{(\alpha)} &= 2\left(e_{14}^{(\alpha)} - e_{16}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{16}^{(\alpha)} - e_{15}^{(\alpha)}\right) \otimes \pi_2, & \tilde{\varepsilon}_2^{(\alpha)} &= 2\left(e_{15}^{(\alpha)} - e_{16}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{14}^{(\alpha)} - e_{15}^{(\alpha)}\right) \otimes \pi_2,\\
\varepsilon_3^{(\alpha)} &= 2\left(e_{44}^{(\alpha)} - e_{66}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{66}^{(\alpha)} - e_{55}^{(\alpha)}\right) \otimes \pi_2, & \tilde{\varepsilon}_3^{(\alpha)} &= 2\left(e_{55}^{(\alpha)} - e_{66}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{44}^{(\alpha)} - e_{55}^{(\alpha)}\right) \otimes \pi_2,\\
\varepsilon_4^{(\alpha)} &= 2\left(e_{45}^{(\alpha)} - e_{64}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{64}^{(\alpha)} - e_{56}^{(\alpha)}\right) \otimes \pi_2, & \tilde{\varepsilon}_4^{(\alpha)} &= 2\left(e_{56}^{(\alpha)} - e_{64}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{45}^{(\alpha)} - e_{56}^{(\alpha)}\right) \otimes \pi_2,\\
\varepsilon_5^{(\alpha)} &= 2\left(e_{46}^{(\alpha)} - e_{65}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{65}^{(\alpha)} - e_{54}^{(\alpha)}\right) \otimes \pi_2, & \tilde{\varepsilon}_5^{(\alpha)} &= 2\left(e_{54}^{(\alpha)} - e_{65}^{(\alpha)}\right) \otimes \pi_1 + 2\left(e_{46}^{(\alpha)} - e_{54}^{(\alpha)}\right) \otimes \pi_2.
\end{aligned} \tag{118}$$

which leads to the intersection form

$$\varepsilon_i^{(\alpha)} \circ \tilde{\varepsilon}_j^{(\beta)} = -4\, \delta_{ij}\, \delta^{\alpha\beta}, \tag{119}$$

and all other intersections vanish.

The relation between a single $\mathbb{Z}_2^{(\alpha)}$ fixed point and the exceptional three-cycles is given in table 33, and the complete assignment for a given bulk cycle, displacement $\vec{\sigma}$ and Wilson line $\vec{\tau}$ is relegated to tables 55 ande 56 in appendix C.

The exceptional three-cycles pick up a sign $-\eta_{(\alpha)}$ under the orientifold projection Ω, while $\mathcal{R}$ permutes the $\mathbb{Z}_2^{(\alpha)}$ fixed points and one-cycles, see figure 2. The result of the orientifold

$\mathbb{Z}_2^{(\alpha)}$ **fixed points and exceptional 3-cycles on** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$	
f.p.$^{(\alpha)} \otimes (n^\alpha \pi_{2\alpha-1} + m^\alpha \pi_{2\alpha})$	orbit
11	$-$
41	$n^\alpha \varepsilon_1^{(\alpha)} + m^\alpha \tilde{\varepsilon}_1^{(\alpha)}$
51	$-(n^\alpha + m^\alpha)\varepsilon_1^{(\alpha)} + n^\alpha \tilde{\varepsilon}_1^{(\alpha)}$
61	$m^\alpha \varepsilon_1^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_1^{(\alpha)}$
14	$n^\alpha \varepsilon_2^{(\alpha)} + m^\alpha \tilde{\varepsilon}_2^{(\alpha)}$
15	$-(n^\alpha + m^\alpha)\varepsilon_2^{(\alpha)} + n^\alpha \tilde{\varepsilon}_2^{(\alpha)}$
16	$m^\alpha \varepsilon_2^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_2^{(\alpha)}$
44	$n^\alpha \varepsilon_3^{(\alpha)} + m^\alpha \tilde{\varepsilon}_3^{(\alpha)}$
45	$n^\alpha \varepsilon_4^{(\alpha)} + m^\alpha \tilde{\varepsilon}_4^{(\alpha)}$
46	$n^\alpha \varepsilon_5^{(\alpha)} + m^\alpha \tilde{\varepsilon}_5^{(\alpha)}$
54	$-(n^\alpha + m^\alpha)\varepsilon_5^{(\alpha)} + n^\alpha \tilde{\varepsilon}_5^{(\alpha)}$
55	$-(n^\alpha + m^\alpha)\varepsilon_3^{(\alpha)} + n^\alpha \tilde{\varepsilon}_3^{(\alpha)}$
56	$-(n^\alpha + m^\alpha)\varepsilon_4^{(\alpha)} + n^\alpha \tilde{\varepsilon}_4^{(\alpha)}$
64	$m^\alpha \varepsilon_4^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_4^{(\alpha)}$
65	$m^\alpha \varepsilon_5^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_5^{(\alpha)}$
66	$m^\alpha \varepsilon_3^{(\alpha)} - (n^\alpha + m^\alpha)\tilde{\varepsilon}_3^{(\alpha)}$

Table 33: Relation between a $\mathbb{Z}_2^{(\alpha)}$ fixed point times a 1-cycle on $T^2_{(\alpha)}$ and the exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$.

projection on exceptional three-cycles from the $\mathbb{Z}_2^{(\alpha)}$ sector is listed in tables 34, 35 and 36 for $\alpha = 1, 2, 3$.

If we write a fractional three-cycle in the form

$$\begin{aligned}
\Pi^{\text{frac}} =& \frac{1}{4}\Pi^{\text{bulk}} + \frac{1}{4}\sum_{\alpha=1}^{3} \Pi^{\mathbb{Z}_2^{(\alpha)}} \\
=& \frac{1}{4}\left(X\rho_1 + Y\rho_2\right) + \frac{1}{4}\sum_{\alpha=1}^{3}\sum_{i=1}^{5}\left(x_i^{(\alpha)}\,\varepsilon_i^{(\alpha)} + y_i^{(\alpha)}\,\tilde{\varepsilon}_i^{(\alpha)}\right),
\end{aligned} \tag{120}$$

$\Omega\mathcal{R}$ on exceptional 3-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_6'$, Part I				
3-cycle	$\varepsilon_i^{(1)}$	$\tilde{\varepsilon}_i^{(1)}$	$i=i'$	$i\leftrightarrow i'$
AAA	$-\eta_{(1)}\,\varepsilon_{i'}^{(1)}$	$\eta_{(1)}\left(\tilde{\varepsilon}_{i'}^{(1)}-\varepsilon_{i'}^{(1)}\right)$	$1,2,3$	$4,5$
AAB	$-\eta_{(1)}\,\varepsilon_{i'}^{(1)}$	$\eta_{(1)}\left(\tilde{\varepsilon}_{i'}^{(1)}-\varepsilon_{i'}^{(1)}\right)$	$1,5$	$3,4$
	$\eta_{(1)}\left(\varepsilon_{2}^{(1)}-\tilde{\varepsilon}_{2}^{(1)}\right)$	$-\eta_{(1)}\,\tilde{\varepsilon}_{2}^{(1)}$	2	$-$
ABB	$\eta_{(1)}\left(\varepsilon_{i'}^{(1)}-\tilde{\varepsilon}_{i'}^{(1)}\right)$	$-\eta_{(1)}\,\tilde{\varepsilon}_{i'}^{(1)}$	$1,2,3$	$4,5$
BBB	$\eta_{(1)}\,\varepsilon_{i'}^{(1)}$	$\eta_{(1)}\left(\varepsilon_{i'}^{(1)}-\tilde{\varepsilon}_{i'}^{(1)}\right)$	$1,2,3$	$4,5$

Table 34: Orientifold projection of the exceptional three-cycles from the $\mathbb{Z}_2^{(1)}$ twisted sector on the $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6'\times\Omega\mathcal{R})$ background. The sign factor is again $\eta_{(1)}\equiv\eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(1)}}$.

$\Omega\mathcal{R}$ on exceptional 3-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_6'$, Part II				
3-cycle	$\varepsilon_i^{(2)}$	$\tilde{\varepsilon}_i^{(2)}$	$i=i'$	$i\leftrightarrow i'$
AAA	$-\eta_{(2)}\,\varepsilon_{i'}^{(2)}$	$\eta_{(2)}\left(\tilde{\varepsilon}_{i'}^{(2)}-\varepsilon_{i'}^{(2)}\right)$	$1,2,3$	$4,5$
AAB	$-\eta_{(2)}\,\varepsilon_{i'}^{(2)}$	$\eta_{(2)}\left(\tilde{\varepsilon}_{i'}^{(2)}-\varepsilon_{i'}^{(2)}\right)$	$1,5$	$3,4$
	$\eta_{(2)}\left(\varepsilon_{2}^{(2)}-\tilde{\varepsilon}_{2}^{(2)}\right)$	$-\eta_{(2)}\,\tilde{\varepsilon}_{2}^{(2)}$	2	$-$
ABB	$-\eta_{(2)}\,\tilde{\varepsilon}_{i'}^{(2)}$	$-\eta_{(2)}\,\varepsilon_{i'}^{(2)}$	$1,5$	$3,4$
	$\eta_{(2)}\,\varepsilon_{2}^{(2)}$	$\eta_{(2)}\left(\varepsilon_{2}^{(2)}-\tilde{\varepsilon}_{2}^{(3)}\right)$	2	$-$
BBB	$\eta_{(2)}\,\varepsilon_{i'}^{(2)}$	$\eta_{(2)}\left(\varepsilon_{i'}^{(2)}-\tilde{\varepsilon}_{i'}^{(2)}\right)$	$1,2,3$	$4,5$

Table 35: Orientifold projection of the exceptional three-cycles from the $\mathbb{Z}_2^{(2)}$ twisted sector on the $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6'\times\Omega\mathcal{R})$ background.

the twisted RR tadpole cancellation conditions take the following form for the four in-

$\Omega\mathcal{R}$ on exceptional 3-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, Part III				
3-cycle	$\varepsilon_i^{(3)}$	$\tilde{\varepsilon}_i^{(3)}$	$i = i'$	$i \leftrightarrow i'$
AAA	$-\eta_{(3)}\,\varepsilon_{i'}^{(3)}$	$\eta_{(3)}\left(\tilde{\varepsilon}_{i'}^{(3)} - \varepsilon_{i'}^{(3)}\right)$	$1,2,3$	$4,5$
AAB	$-\eta_{(3)}\,\tilde{\varepsilon}_{i'}^{(3)}$	$-\eta_{(3)}\,\varepsilon_{i'}^{(3)}$	$1,2,3$	$4,5$
ABB	$-\eta_{(3)}\,\tilde{\varepsilon}_{i'}^{(3)}$	$-\eta_{(3)}\,\varepsilon_{i'}^{(3)}$	$1,5$	$3,4$
	$\eta_{(3)}\,\varepsilon_2^{(3)}$	$\eta_{(3)}\left(\varepsilon_2^{(3)} - \tilde{\varepsilon}_2^{(3)}\right)$	2	$-$
BBB	$\eta_{(3)}\,\varepsilon_{i'}^{(3)}$	$\eta_{(3)}\left(\varepsilon_{i'}^{(3)} - \tilde{\varepsilon}_{i'}^{(3)}\right)$	$1,2,3$	$4,5$

Table 36: Orientifold projection of the exceptional three-cycles from the $\mathbb{Z}_2^{(3)}$ twisted sector on the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ background.

equivalent lattices,

$$
\begin{aligned}
&\mathbf{AAA}:\\
&\sum_{\alpha=1}^{3}\sum_{i=1}^{3}\left(\sum_a N_a[(1-\eta_{(\alpha)})\,x_{i,a}^{(\alpha)} - \eta_{(\alpha)}\,y_{i,a}^{(\alpha)}]\right)\varepsilon_i^{(\alpha)} + \left(\sum_a N_a(1+\eta_{(\alpha)})\,y_{i,a}^{(\alpha)}\right)\tilde{\varepsilon}_i^{(\alpha)} =\\
&-\sum_{\alpha=1}^{3}\left(\sum_a N_a[x_{4,a}^{(\alpha)} - \eta_{(\alpha)}x_{5,a}^{(\alpha)} + \frac{1}{2}(y_{4,a}^{(\alpha)} - \eta_{(\alpha)}y_{5,a}^{(\alpha)})]\right)(\varepsilon_4^{(\alpha)} - \eta_{(\alpha)}\varepsilon_5^{(\alpha)})\\
&-\sum_{\alpha=1}^{3}\left(\sum_a N_a(y_{4,a}^{(\alpha)} + \eta_{(\alpha)}y_{5,a}^{(\alpha)})\right)[\tilde{\varepsilon}_4^{(\alpha)} + \eta_{(\alpha)}\,\tilde{\varepsilon}_5^{(\alpha)} - \frac{1}{2}(\varepsilon_4^{(\alpha)} + \eta_{(1)}\varepsilon_5^{(\alpha)})],
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{AAB}:\\
&\sum_{\alpha=1}^{2}\sum_{i=1,5}\left\{\left(\sum_a N_a[(1-\eta_{(\alpha)})\,x_{i,a}^{(\alpha)} - \eta_{(\alpha)}\,y_{i,a}^{(\alpha)}]\right)\varepsilon_i^{(\alpha)} + \left(\sum_a N_a(1+\eta_{(\alpha)})\,y_{i,a}^{(\alpha)}\right)\tilde{\varepsilon}_i^{(\alpha)}\right\} =\\
&-\sum_{\alpha=1}^{2}\left(\sum_a N_a[x_{3,a}^{(\alpha)} - \eta_{(\alpha)}x_{4,a}^{(\alpha)} + \frac{1}{2}(y_{3,a}^{(\alpha)} - \eta_{(\alpha)}y_{4,a}^{(\alpha)})]\right)(\varepsilon_3^{(\alpha)} - \eta_{(\alpha)}\varepsilon_4^{(\alpha)})\\
&-\sum_{\alpha=1}^{2}\left(\sum_a N_a(y_{3,a}^{(\alpha)} + \eta_{(\alpha)}y_{4,a}^{(\alpha)})\right)[\tilde{\varepsilon}_3^{(\alpha)} + \eta_{(\alpha)}\,\tilde{\varepsilon}_4^{(\alpha)} - \frac{1}{2}(\varepsilon_3^{(\alpha)} + \eta_{(\alpha)}\varepsilon_4^{(\alpha)})]\\
&-\sum_{\alpha=1}^{2}\left\{\left(\sum_a N_a(1+\eta_{(\alpha)})x_{2,a}^{(\alpha)}\right)\varepsilon_2^{(\alpha)} - \left(\sum_a N_a[-\eta_{(1)}\,x_{2,a}^{(\alpha)} + (1-\eta_{(\alpha)})y_{2,a}^{(\alpha)}]\right)\tilde{\varepsilon}_2^{(\alpha)}\right\},
\end{aligned}
$$

$$\sum_{i=1}^{3}\left(\sum_a N_a(x^{(3)}_{i,a}-\eta_{(3)}y^{(3)}_{i,a})\right)\,(\varepsilon^{(3)}_i-\eta_{(3)}\tilde{\varepsilon}^{(3)}_i)=$$
$$-\left(\sum_a N_a(x^{(3)}_{4,a}-\eta_{(3)}y^{(3)}_{5,a})\right)(\varepsilon^{(3)}_4-\eta_{(3)}\tilde{\varepsilon}^{(3)}_5)$$
$$-\left(\sum_a N_a(x^{(3)}_{5,a}-\eta_{(3)}y^{(3)}_{4,a})\right)(\varepsilon^{(3)}_5-\eta_{(3)}\tilde{\varepsilon}^{(3)}_4),$$

ABB :

$$\sum_{i=1}^{3}\left\{\left(\sum_a N_a(1+\eta_{(1)})\,x^{(1)}_{i,a}\right)\,\varepsilon^{(1)}_i+\left(\sum_a N_a[-\eta_{(1)}\,x^{(1)}_{i,a}+(1-\eta_{(1)})\,y^{(1)}_{i,a}]\right)\tilde{\varepsilon}^{(1)}_i\right\}=$$
$$-\left(\sum_a N_a(x^{(1)}_{4,a}+\eta_{(1)}x^{(1)}_{5,a})\right)\,[\varepsilon^{(1)}_4+\eta_{(1)}\varepsilon^{(1)}_5-\frac{1}{2}(\tilde{\varepsilon}^{(1)}_4+\eta_{(1)}\tilde{\varepsilon}^{(1)}_5)]$$
$$-\left(\sum_a N_a[y^{(1)}_{4,a}-\eta_{(1)}y^{(1)}_{5,a}+\frac{1}{2}(x^{(1)}_{4,a}-\eta_{(1)}x^{(1)}_{5,a})]\right)\,(\tilde{\varepsilon}^{(1)}_4-\eta_{(1)}\tilde{\varepsilon}^{(1)}_5),$$
$$\sum_{\alpha=2}^{3}\sum_{i=1,5}\left(\sum_a N_a(x^{(\alpha)}_{i,a}-\eta_{(\alpha)}y^{(\alpha)}_{i,a})\right)\,(\varepsilon^{(\alpha)}_i-\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_i)=$$
$$-\sum_{\alpha=2}^{3}\left(\sum_a N_a(x^{(\alpha)}_{3,a}-\eta_{(\alpha)}y^{(\alpha)}_{4,a})\right)(\varepsilon^{(\alpha)}_3-\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_4)$$
$$-\sum_{\alpha=2}^{3}\left(\sum_a N_a(x^{(\alpha)}_{4,a}-\eta_{(\alpha)}y^{(\alpha)}_{3,a})\right)(\varepsilon^{(\alpha)}_4-\eta_{(\alpha)}\tilde{\varepsilon}^{(\alpha)}_3)$$
$$-\sum_{\alpha=2}^{3}\left\{\left(\sum_a N_a[(1+\eta_{(\alpha)})\,x^{(\alpha)}_{2,a}+\eta_{(\alpha)}\,y^{(\alpha)}_{2,a}]\right)\varepsilon^{(\alpha)}_2-\left(\sum_a N_a(1-\eta_{(\alpha)})\,y^{(\alpha)}_{2,a}\right)\,\tilde{\varepsilon}^{(\alpha)}_2\right\},$$

BBB :

$$\sum_{\alpha=1}^{3}\sum_{i=1}^{3}\left(\sum_a N_a[(1+\eta_{(\alpha)})\,x^{(\alpha)}_{i,a}+\eta_{(\alpha)}\,y^{(\alpha)}_{i,a}]\right)\varepsilon^{(\alpha)}_i+\left(\sum_a N_a(1-\eta_{(\alpha)})\,y^{(\alpha)}_{i,a}\right)\,\tilde{\varepsilon}^{(\alpha)}_i=$$
$$-\sum_{\alpha=1}^{3}\left(\sum_a N_a[x^{(\alpha)}_{4,a}+\eta_{(\alpha)}x^{(\alpha)}_{5,a}+\frac{1}{2}(y^{(\alpha)}_{4,a}+\eta_{(\alpha)}y^{(\alpha)}_{5,a})]\right)\,(\varepsilon^{(\alpha)}_4+\eta_{(\alpha)}\varepsilon^{(\alpha)}_5)$$
$$-\sum_{\alpha=1}^{3}\left(\sum_a N_a(y^{(\alpha)}_{4,a}-\eta_{(\alpha)}y^{(\alpha)}_{5,a})\right)\,[\tilde{\varepsilon}^{(\alpha)}_4-\eta_{(\alpha)}\,\tilde{\varepsilon}^{(\alpha)}_5-\frac{1}{2}(\varepsilon^{(\alpha)}_4-\eta_{(1)}\varepsilon^{(\alpha)}_5)].$$

Notice that the twisted RR tadpole conditions on **AAA** and **BBB** are related by replacing $\eta_{(\alpha)}\to-\eta_{(\alpha)}$.

The integer coefficients $(x_i^{(\alpha)}, y_i^{(\alpha)})$ can be read off from tables 55 and 56 in appendix C. Per twist sector and D6-brane, exactly three of the five pairs are non-vanishing. Each of the pairs originates from either a single fixed point (I) or two fixed point contributions (II). Their shape is restricted to the following six options,

(x, y)	
I.	II.
$\pm(n, m)$	$\pm(-n + [z-1]\,m, -z\,m + [1-z]\,n)$
$\pm(-n-m, n)$	$\pm(n + z\,m, -z\,n + [1-z]\,m)$
$\pm(m, -n-m)$	$\pm([1-z]\,n - z\,m, z\,n + m)$

with $z \in \{(-1)^{\tau_j}, (-1)^{\tau_k}, (-1)^{\tau_j+\tau_k}\}$ in the $\mathbb{Z}_2^{(i)}$ twisted sector. The global signs depend also on these Wilson lines, as well as the $\mathbb{Z}_2^{(i)}$ eigenvalue $\tau_0^{(i)}$. Note that this shape is very similar to the one for the $\mathbb{Z}_2^{(2)}$ and $\mathbb{Z}_2^{(3)}$ sectors on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ with discrete torsion discussed in section 5.2.

The knowledge of the general form of the coefficients in the twisted sector contributions is expected to be useful when simplifying the K-theory constraints in the following section.

6.3 K-theory constraint

The classification of $\Omega\mathcal{R}$ invariant D6-branes follows again closely the one for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ in section 3.3. Possible probe D6-branes are of the form (52), where the bulk contributions can be read off from table 32, and the exceptional contributions are listed in tables 37 to 39 for the four inequivalent lattice orientations. The existence of $\Omega\mathcal{R}$ invariant D6-branes boils again down to the relations in table 10 among the sets of signs in the orientifold projection $\{\eta_{(i)}\}$ and discrete displacements $\vec{\sigma}$ and Wilson lines $\vec{\tau}$, where $b_1 = b_2 = b_3 = \frac{1}{2}$ due to the $SU(3)^3$ compactifaction lattice. The K-theory constraints can be explicitly derived by using the intersection numbers,

$$
\begin{aligned}
\Pi_a^{\mathrm{frac}} \circ \Pi_b^{\mathrm{frac}} = & \frac{1}{4}\left(X_a Y_b - Y_a X_b - \sum_{\alpha=1}^{3} \left(\vec{x}_a^{(\alpha)} \cdot \vec{y}_b^{(\alpha)} - \vec{x}_b^{(\alpha)} \cdot \vec{y}_a^{(\alpha)} \right) \right) \\
& \text{with} \quad \vec{x}_a^{(\alpha)} \cdot \vec{y}_b^{(\alpha)} \equiv \sum_{m=1}^{5} x_{m,a}^{(\alpha)} y_{m,b}^{(\alpha)},
\end{aligned}
\tag{121}
$$

when the expansion (120) of a fractional three-cycle on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ is used.

The bulk contributions to the K-theory constraints are given in table 40, which after RR tadpole cancellation simplify to the expression in table 41. The K-theory constraints

Exceptional contributions to $\Omega\mathcal{R}$ invariant D6-branes on $T^6/\mathbb{Z}_2\times\mathbb{Z}_6'$, Part I	
AAA	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,0)}$	$(-1)^{\tau_j}\varepsilon_1^{(i)}+(-1)^{\tau_k}\varepsilon_2^{(i)}+(-1)^{\tau_j+\tau_k}\varepsilon_3^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,0)}$	$-\varepsilon_1^{(i)}+[1-(-1)^{\tau_j}]\tilde{\varepsilon}_1^{(i)}-(-1)^{\tau_j+\tau_k}\tilde{\varepsilon}_4^{(i)}+(-1)^{\tau_k}[-\varepsilon_5^{(i)}+\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,1)}$	$-\varepsilon_2^{(i)}+(1-(-1)^{\tau_k})\tilde{\varepsilon}_2^{(i)}+(-1)^{\tau_j}\varepsilon_4^{(i)}+(-1)^{\tau_j+\tau_k}\varepsilon_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,1)}$	$-\varepsilon_3^{(i)}+(1-(-1)^{\tau_j+\tau_k})\tilde{\varepsilon}_3^{(i)}+(-1)^{\tau_k}[-\varepsilon_4^{(i)}+\tilde{\varepsilon}_4^{(i)}]-(-1)^{\tau_j}\tilde{\varepsilon}_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,0)}$	$(-1)^{\tau_j}[-\varepsilon_1^{(i)}+2\tilde{\varepsilon}_1^{(i)}]+(-1)^{\tau_k}[-\varepsilon_2^{(i)}+2\tilde{\varepsilon}_2^{(i)}]+(-1)^{\tau_j+\tau_k}[-\varepsilon_3^{(i)}+2\tilde{\varepsilon}_3^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,0)}$	$[2(-1)^{\tau_j}-1)]\varepsilon_1^{(i)}+[-1-(-1)^{\tau_j}]\tilde{\varepsilon}_1^{(i)}+(-1)^{\tau_j+\tau_k}[2\varepsilon_4^{(i)}-\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_k}[-\varepsilon_5^{(i)}-\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,1)}$	$[-1+2(-1)^{\tau_k}]\varepsilon_2^{(i)}-[1+(-1)^{\tau_k}]\tilde{\varepsilon}_2^{(i)}+(-1)^{\tau_j}[-\varepsilon_4^{(i)}+2\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_j+\tau_k}[-\varepsilon_5^{(i)}+2\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,1)}$	$[-1+2(-1)^{\tau_j+\tau_k}]\varepsilon_3^{(i)}-[1+(-1)^{\tau_j+\tau_k}]\tilde{\varepsilon}_3^{(i)}-(-1)^{\tau_k}[\varepsilon_4^{(i)}+\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_j}[2\varepsilon_5^{(i)}-\tilde{\varepsilon}_5^{(i)}]$
BBB	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,0)}$	$(-1)^{\tau_j}\left[\varepsilon_1^{(i)}-2\tilde{\varepsilon}_1^{(i)}\right]+(-1)^{\tau_k}\left[\varepsilon_2^{(i)}-2\tilde{\varepsilon}_2^{(i)}\right]+(-1)^{\tau_j+\tau_k}\left[\varepsilon_3^{(i)}-2\tilde{\varepsilon}_3^{(i)}\right]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,0)}$	$(1-2(-1)^{\tau_j})\varepsilon_1^{(i)}+(1+(-1)^{\tau_j})\tilde{\varepsilon}_1^{(i)}+(-1)^{\tau_j+\tau_k}[-2\varepsilon_4^{(i)}+\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_k}[\varepsilon_5^{(i)}+\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,1)}$	$(1-2(-1)^{\tau_k})\varepsilon_2^{(i)}+(1+(-1)^{\tau_k})\tilde{\varepsilon}_2^{(i)}+(-1)^{\tau_j}[\varepsilon_4^{(i)}-2\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_j+\tau_k}[\varepsilon_5^{(i)}-2\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,1)}$	$(1-2(-1)^{\tau_j+\tau_k})\varepsilon_3^{(i)}+(1+(-1)^{\tau_j+\tau_k})\tilde{\varepsilon}_3^{(i)}+(-1)^{\tau_k}[\varepsilon_4^{(i)}+\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_j}[-2\varepsilon_5^{(i)}+\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,0)}$	$-(-1)^{\tau_j}\varepsilon_1^{(i)}-(-1)^{\tau_k}\varepsilon_2^{(i)}-(-1)^{\tau_j+\tau_k}\varepsilon_3^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,0)}$	$\varepsilon_1^{(i)}+(-1+(-1)^{\tau_j})\tilde{\varepsilon}_1^{(i)}+(-1)^{\tau_j+\tau_k}\tilde{\varepsilon}_4^{(i)}+(-1)^{\tau_k}[\varepsilon_5^{(i)}-\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,1)}$	$\varepsilon_2^{(i)}+(-1+(-1)^{\tau_k})\tilde{\varepsilon}_2^{(i)}-(-1)^{\tau_j}\varepsilon_4^{(i)}-(-1)^{\tau_j+\tau_k}\varepsilon_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,1)}$	$\varepsilon_3^{(i)}+(-1+(-1)^{\tau_j+\tau_k})\tilde{\varepsilon}_3^{(i)}+(-1)^{\tau_k}[\varepsilon_4^{(i)}-\tilde{\varepsilon}_4^{(i)}]+(-1)^{\tau_j}\tilde{\varepsilon}_5^{(i)}$

Table 37: Exceptional three-cycles for $\Omega\mathcal{R}$ invariant D6-branes, which contribute to the K-theory constraint on the **AAA** and **BBB** lattices on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6'\times\Omega\mathcal{R})$ with discrete torsion. All cycles habe to be multiplied by the $\mathbb{Z}_2^{(i)}$ eigenvalues $(-1)^{\tau_0^{(i)}}$. The subscript h labels 1-cycles parallel to the $\Omega\mathcal{R}$ plane, cycles with subscript v have the 1-cycle contribution perpendicular to the $\Omega\mathcal{R}$ plane. The discrete displacements on the remaining four-torus are given in parenthesis. Notice the symmetry **AAA** $:(\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_j,\sigma_k)},\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_j,\sigma_k)})\Leftrightarrow$ **BBB** $:(-\Pi^{\mathbb{Z}_2^{(i)}}_{v,(\sigma_j,\sigma_k)},-\Pi^{\mathbb{Z}_2^{(i)}}_{h,(\sigma_j,\sigma_k)})$ for the same choice of discrete displacements (σ_j,σ_k).

on $T^6/(\mathbb{Z}_2\times\mathbb{Z}_6'\times\Omega\mathcal{R})$ without discrete torsion are thus trivially fulfilled, if RR tadpoles are cancelled. In the case with discrete torsion, the exceptional contributions have to be

Exceptional contributions to $\Omega\mathcal{R}$ invariant D6-branes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, Part II	
$\mathbf{AAB}$: $\mathbb{Z}_2^{(i)}$ with $i = 1, 2$	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,0)}$	$(-1)^{\tau_j}\varepsilon_1^{(i)} - (-1)^{\tau_3}\tilde{\varepsilon}_2^{(i)} + (-1)^{\tau_j+\tau_3}\varepsilon_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,0)}$	$-\varepsilon_1^{(i)} + [1 - (-1)^{\tau_j}]\tilde{\varepsilon}_1^{(i)} - (-1)^{\tau_j+\tau_3}\tilde{\varepsilon}_3^{(i)} + (-1)^{\tau_3}[-\varepsilon_4^{(i)} + \tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,1)}$	$(1 - (-1)^{\tau_3})\varepsilon_2^{(i)} + (-1)^{\tau_3}\tilde{\varepsilon}_2^{(i)} + (-1)^{\tau_j}\varepsilon_3^{(i)} + (-1)^{\tau_j+\tau_3}\varepsilon_4^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,1)}$	$(-1)^{\tau_3}[-\varepsilon_3^{(i)} + \tilde{\varepsilon}_3^{(i)}] - (-1)^{\tau_j}\tilde{\varepsilon}_4^{(i)} - \varepsilon_5^{(i)} + [1 - (-1)^{\tau_j+\tau_3}]\tilde{\varepsilon}_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,0)}$	$(-1)^{\tau_j}[-\varepsilon_1^{(i)} + 2\tilde{\varepsilon}_1^{(i)}] + (-1)^{\tau_3}[2\varepsilon_2^{(i)} - \tilde{\varepsilon}_2^{(i)}] + (-1)^{\tau_j+\tau_3}[-\varepsilon_5^{(i)} + 2\tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,0)}$	$[2(-1)^{\tau_j} - 1]\varepsilon_1^{(i)} - [(-1)^{\tau_j} + 1]\tilde{\varepsilon}_1^{(i)} + (-1)^{\tau_j+\tau_3}[2\varepsilon_3^{(i)} - \tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_3}[-\varepsilon_4^{(i)} - \tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,1)}$	$-(1 + (-1)^{\tau_3})\varepsilon_2^{(i)} + (2 - (-1)^{\tau_3})\tilde{\varepsilon}_2^{(i)} + (-1)^{\tau_j}[-\varepsilon_3^{(i)} + 2\tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_j+\tau_3}[-\varepsilon_4^{(i)} + 2\tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,1)}$	$-(-1)^{\tau_3}[\varepsilon_3^{(i)} + \tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_j}[2\varepsilon_4^{(i)} - \tilde{\varepsilon}_4^{(i)}][2(-1)^{\tau_j+\tau_3} - 1]\varepsilon_5^{(i)} - [1 + (-1)^{\tau_j+\tau_3}]\tilde{\varepsilon}_5^{(i)}$
$\mathbf{AAB}$: $\mathbb{Z}_2^{(3)}$	
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(0,0)}$	$(-1)^{\tau_1}[\varepsilon_1^{(3)} + \tilde{\varepsilon}_1^{(3)}] + (-1)^{\tau_2}[\varepsilon_2^{(3)} + \tilde{\varepsilon}_2^{(3)}] + (-1)^{\tau_1+\tau_2}[\varepsilon_3^{(3)} + \tilde{\varepsilon}_3^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(1,0)}$	$[(-1)^{\tau_1} - 2]\varepsilon_1^{(3)} + [1 - 2(-1)^{\tau_1}]\tilde{\varepsilon}_1^{(3)} + (-1)^{\tau_1+\tau_2}[\varepsilon_4^{(3)} - 2\tilde{\varepsilon}_4^{(3)}] + (-1)^{\tau_2}[-2\varepsilon_5^{(3)} + \tilde{\varepsilon}_5^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(0,1)}$	$[(-1)^{\tau_2} - 2]\varepsilon_2^{(3)} + [1 - 2(-1)^{\tau_2}]\tilde{\varepsilon}_2^{(3)} + (-1)^{\tau_1}[\varepsilon_4^{(3)} + \tilde{\varepsilon}_4^{(3)}] + (-1)^{\tau_1+\tau_2}[\varepsilon_5^{(i)} + \tilde{\varepsilon}_5^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{h,(1,1)}$	$[(-1)^{\tau_1+\tau_2} - 2]\varepsilon_3^{(3)} + [1 - 2(-1)^{\tau_1+\tau_2}]\tilde{\varepsilon}_3^{(3)} + (-1)^{\tau_2}[-2\varepsilon_4^{(3)} + \tilde{\varepsilon}_4^{(3)}] + (-1)^{\tau_1}[\varepsilon_5^{(3)} - 2\tilde{\varepsilon}_5^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(0,0)}$	$(-1)^{\tau_1}[\varepsilon_1^{(3)} - \tilde{\varepsilon}_1^{(3)}] + (-1)^{\tau_2}[\varepsilon_2^{(3)} - \tilde{\varepsilon}_2^{(3)}] + (-1)^{\tau_1+\tau_2}[\varepsilon_3^{(3)} - \tilde{\varepsilon}_3^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(1,0)}$	$-(-1)^{\tau_1}\varepsilon_1^{(3)} + \tilde{\varepsilon}_1^{(3)} - (-1)^{\tau_1+\tau_2}\varepsilon_4^{(3)} + (-1)^{\tau_2}\tilde{\varepsilon}_5^{(3)}$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(0,1)}$	$-(-1)^{\tau_2}\varepsilon_2^{(3)} + \tilde{\varepsilon}_2^{(3)} + (-1)^{\tau_1}[\varepsilon_4^{(3)} - \tilde{\varepsilon}_4^{(3)}] + (-1)^{\tau_1+\tau_2}[\varepsilon_5^{(i)} - \tilde{\varepsilon}_5^{(3)}]$
$\Pi^{\mathbb{Z}_2^{(3)}}_{v,(1,1)}$	$-(-1)^{\tau_1+\tau_2}\varepsilon_3^{(3)} + \tilde{\varepsilon}_3^{(3)} + (-1)^{\tau_2}\tilde{\varepsilon}_4^{(3)} - (-1)^{\tau_1}\varepsilon_5^{(3)}$

Table 38: Exceptional three-cycles for $\Omega\mathcal{R}$ invariant D6-branes, which contribute to the K-theory constraint on the **AAB** lattice on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ with discrete torsion. All cycles habe to be multiplied by the $\mathbb{Z}_2^{(i)}$ eigenvalues $(-1)^{\tau_0^{(i)}}$. For more details on the notation see table 37.

evaluated explicitly. Due to the large number of combinatorial possibilities displayed in tables 37 to 39, we do not write out all constraints, but we expect that also the twisted contributions to the K-theory constraints can be simplified due to the shape of the coefficients (x, y) presented at the end of section 6.2. The explicit examples below fulfill the K-theory constraint trivially by only having gauge groups of even rank.

Exceptional contributions to $\Omega\mathcal{R}$ invariant D6-branes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, Part III	
ABB : $\mathbb{Z}_2^{(1)}$	
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,0)}$	$-(-1)^{\tau_2}\tilde{\varepsilon}_1^{(1)} - (-1)^{\tau_3}\tilde{\varepsilon}_2^{(1)} - (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,0)}$	$(1-(-1)^{\tau_2})\varepsilon_1^{(1)} + (-1)^{\tau_2}\tilde{\varepsilon}_1^{(1)} + (-1)^{\tau_2+\tau_3}[-\varepsilon_4^{(1)} + \tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_3}\varepsilon_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(0,1)}$	$(1-(-1)^{\tau_3})\varepsilon_2^{(1)} + (-1)^{\tau_3}\tilde{\varepsilon}_2^{(1)} - (-1)^{\tau_2}\tilde{\varepsilon}_4^{(1)} - (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_5^{(1)}$
$\Pi^{\mathbb{Z}_2^{(1)}}_{h,(1,1)}$	$(1-(-1)^{\tau_2+\tau_3})\varepsilon_3^{(1)} + (-1)^{\tau_2+\tau_3}\tilde{\varepsilon}_3^{(1)} + (-1)^{\tau_3}\varepsilon_4^{(1)} + (-1)^{\tau_2}[-\varepsilon_5^{(1)} + \tilde{\varepsilon}_5^{(1)}]$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,0)}$	$(-1)^{\tau_2}\left[2\varepsilon_1^{(1)} - \tilde{\varepsilon}_1^{(1)}\right] + (-1)^{\tau_3}\left[2\varepsilon_2^{(1)} - \tilde{\varepsilon}_2^{(1)}\right] + (-1)^{\tau_2+\tau_3}\left[2\varepsilon_3^{(1)} - \tilde{\varepsilon}_3^{(1)}\right]$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,0)}$	$-[1+(-1)^{\tau_2}]\varepsilon_1^{(1)} + (2-(-1)^{\tau_2})\tilde{\varepsilon}_1^{(1)} - (-1)^{\tau_2+\tau_3}[\varepsilon_4^{(1)} + \tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_3}[-\varepsilon_5^{(1)} + 2\tilde{\varepsilon}_5^{(1)}]$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(0,1)}$	$-[1+(-1)^{\tau_3}]\varepsilon_2^{(1)} + (2-(-1)^{\tau_3})\tilde{\varepsilon}_2^{(1)} + (-1)^{\tau_2}[2\varepsilon_4^{(1)} - \tilde{\varepsilon}_4^{(1)}] + (-1)^{\tau_2+\tau_3}[2\varepsilon_5^{(1)} - \tilde{\varepsilon}_5^{(1)}]$
$\Pi^{\mathbb{Z}_2^{(1)}}_{v,(1,1)}$	$-[1+(-1)^{\tau_2+\tau_3}]\varepsilon_3^{(1)} + (2-(-1)^{\tau_2+\tau_3})\tilde{\varepsilon}_3^{(1)} + (-1)^{\tau_3}[-\varepsilon_4^{(1)} + 2\tilde{\varepsilon}_4^{(1)}] - (-1)^{\tau_2}[\varepsilon_5^{(1)} + \tilde{\varepsilon}_5^{(1)}]$
ABB : $\mathbb{Z}_2^{(i)}$ with $i=2,3$	
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,0)}$	$(-1)^{\tau_1}[\varepsilon_1^{(i)} + \tilde{\varepsilon}_1^{(i)}] + (-1)^{\tau_k}[\varepsilon_2^{(i)} - 2\tilde{\varepsilon}_2^{(i)}] + (-1)^{\tau_1+\tau_k}[\varepsilon_5^{(i)} + \tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,0)}$	$[(-1)^{\tau_1} - 2]\varepsilon_1^{(i)} + [1-2(-1)^{\tau_1}]\tilde{\varepsilon}_1^{(i)} + (-1)^{\tau_1+\tau_k}[\varepsilon_3^{(i)} - 2\tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_k}[-2\varepsilon_4^{(i)} + \tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(0,1)}$	$(1-2(-1)^{\tau_k})\varepsilon_2^{(i)} + (1+(-1)^{\tau_k})\tilde{\varepsilon}_2^{(i)} + (-1)^{\tau_1}[\varepsilon_3^{(i)} + \tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_1+\tau_k}[\varepsilon_4^{(i)} + \tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{h,(1,1)}$	$(-1)^{\tau_k}[-2\varepsilon_3^{(i)} + \tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_1}[\varepsilon_4^{(i)} - 2\tilde{\varepsilon}_4^{(i)}] + [(-1)^{\tau_1+\tau_k} - 2]\varepsilon_5^{(i)} + [(1-2(-1)^{\tau_1+\tau_k}]\tilde{\varepsilon}_5^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,0)}$	$(-1)^{\tau_1}[\varepsilon_1^{(i)} - \tilde{\varepsilon}_1^{(i)}] - (-1)^{\tau_k}\varepsilon_2^{(i)} + (-1)^{\tau_1+\tau_k}[\varepsilon_5^{(i)} - \tilde{\varepsilon}_5^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,0)}$	$-(-1)^{\tau_1}\varepsilon_1^{(i)} + \tilde{\varepsilon}_1^{(i)} - (-1)^{\tau_1+\tau_k}\varepsilon_3^{(i)} + (-1)^{\tau_k}\tilde{\varepsilon}_4^{(i)}$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(0,1)}$	$\varepsilon_2^{(i)} + ((-1)^{\tau_k} - 1)\tilde{\varepsilon}_2^{(i)} + (-1)^{\tau_1}[\varepsilon_3^{(i)} - \tilde{\varepsilon}_3^{(i)}] + (-1)^{\tau_1+\tau_k}[\varepsilon_4^{(i)} - \tilde{\varepsilon}_4^{(i)}]$
$\Pi^{\mathbb{Z}_2^{(i)}}_{v,(1,1)}$	$(-1)^{\tau_k}\tilde{\varepsilon}_3^{(i)} - (-1)^{\tau_1}\varepsilon_4^{(i)} - (-1)^{\tau_1+\tau_k}\varepsilon_5^{(i)} + \tilde{\varepsilon}_5^{(i)}$

Table 39: Exceptional three-cycles for $\Omega\mathcal{R}$ invariant D6-branes, which contribute to the K-theory constraint on the **ABB** lattice on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ with discrete torsion. All cycles habe to be multiplied by the $\mathbb{Z}_2^{(i)}$ eigenvalues $(-1)^{\tau_0^{(i)}}$. For more details on the notation see table 37.

The discussion in appendix B.2 confirms again that all D6-branes in the classification of $\Omega\mathcal{R}$ cycles carry indeed $Sp(2M)$ gauge groups.

Bulk part of the K-theory constraints on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$					
		AAA	**AAB**	**ABB**	**BBB**
$\Omega\mathcal{R}$	$\frac{1}{2^{3-\eta}} \sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_0}^{\text{bulk}}$	$\frac{-1}{2^{1-\eta}} \sum_a N_a Y_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (X_a - Y_a)$	$\frac{3}{2^{1-\eta}} \sum_a N_a X_a$	$\frac{3}{2^{1-\eta}} \sum_a N_a (2X_a + Y_a)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$\frac{1}{2^{3-\eta}} \sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_1}^{\text{bulk}}$	$\frac{-3}{2^{1-\eta}} \sum_a N_a Y_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (X_a - Y_a)$	$\frac{1}{2^{1-\eta}} \sum_a N_a X_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (2X_a + Y_a)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$\frac{1}{2^{3-\eta}} \sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_2}^{\text{bulk}}$	$\frac{-3}{2^{1-\eta}} \sum_a N_a Y_a$	$\frac{3}{2^{1-\eta}} \sum_a N_a (X_a - Y_a)$	$\frac{3}{2^{1-\eta}} \sum_a N_a X_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (2X_a + Y_a)$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$\frac{1}{2^{3-\eta}} \sum_a N_a \Pi_a^{\text{bulk}} \circ \Pi_{Sp(2)_3}^{\text{bulk}}$	$\frac{-3}{2^{1-\eta}} \sum_a N_a Y_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (X_a - Y_a)$	$\frac{3}{2^{1-\eta}} \sum_a N_a X_a$	$\frac{1}{2^{1-\eta}} \sum_a N_a (2X_a + Y_a)$

Table 40: Bulk contributions to the K-theory constraints on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ without ($\eta = 1$) and with ($\eta = -1$) discrete torsion. The sums can be simplified using the bulk RR tadpole cancellation conditions (117). The result is displayed in table 41.

Simplified contributions to the K-theory constraints on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$							
AAA	after RR tcc mod 2	**AAB**	after RR tcc mod 2	**ABB**	after RR tcc mod 2	**BBB**	after RR tcc mod 2
$\Omega\mathcal{R}$	$-2^\eta \sum_a N_a X_a$	$\Omega\mathcal{R}$	$2^\eta \sum_a N_a Y_a$	$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$-2^\eta \sum_a N_a Y_a$	$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$2^\eta \sum_a N_a X_a$

Table 41: Simplified bulk contributions to the K-theory constraints upon RR tadpole cancellation (117). The other contributions are given by integer multiples of the listed ones, as shown in table 40.

6.4 A $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ example without discrete torsion

In [70], a supersymmetric model with fractional D6-branes on the **AAB** lattice on the orbifold $T^6/\mathbb{Z}_6$ was presented. After the choice of a representant per orbifold and orientifold orbit as discussed at the end of section 6.1, the torus cycles are those listed in table 42. Combining the untwisted RR tadpole cancellation condition (117) with the supersymmetry constraint (116) on the **AAB** lattice leads to

$$\sum_a N_a X_a = 6 \tag{122}$$

without discrete torsion. This equation can be solved for brane a in table 42 with $N_a = 6$, i.e.

$$\Pi_a = \frac{1}{2} (\rho_1 + \rho_2) \, . \tag{123}$$

Supersymmetric bulk 3-cycles on the AAB lattice on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$				
brane	$\frac{\text{angle}}{\pi}$	$(n^1, m^1; n^2, m^2; n^3, m^3)$	X	Y
a, b, d	$(\frac{1}{3}, -\frac{1}{3}, 0)$	$(0, 1; 1, -1; 1, 1)$	1	1
$c, e/\Omega\mathcal{R}$	$(0, 0, 0)$	$(1, 0; 1, 0; 1, 1)$	1	1
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$(\frac{1}{2}, -\frac{1}{2}, 0)$	$(-1, 2; 1, -2; 1, 1)$	3	3
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$(0, \frac{1}{2}, -\frac{1}{2})$	$(1, 0; -1, 2; 1, -1)$	1	1
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(-1, 2; 1, 0; 1, -1)$	1	1

Table 42: Some supersymmetric bulk cycles on the **AAB** on the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ background. The names $a \dots e$ are taken from [70]. $\Omega\mathcal{R}$ labels a cycle which is parallel to the first O6-plane orbit.

The gauge group is thus $U(6)$, where in contrast to the earlier examples on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, the Abelian subgroup is anomaly-free and stays massless. The massless matter spectrum is completely non-chiral and can be computed along the lines described in appendix B.2 by using the torus intersection numbers $I_{a(\omega^k a)}$, $I_{a(\omega^k a')}$ and $I_{a,\Omega\mathcal{R}\theta^n\omega^m}$. This leads to

- three multiplets in the adjoint representation of $U(6)$ from the aa sector,
- three adjoints from the $a(\omega^k a)_{k=1,2}$ sectors,
- one non-chiral pair of antisymmetric representations from the aa' sector,
- five non-chiral pairs of antisymmetrics from the $a(\omega a') + a(\omega^2 a')$ sectors,
- one non-chiral pair of symmetrics from the $a(\omega a') + a(\omega^2 a')$ sectors.

The total massless spectrum is thus completely non-chiral and consists thus of six multiplets in the adjoint representation of $U(6)$, six non-chiral pairs of antisymmetric representations and one non-chiral pair of symmetric representations.

The K-theory constraints are trivially fulfilled due to the even rank of the single gauge factor $U(6)$.

6.5 A $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_6' \times \Omega\mathcal{R})$ example with discrete torsion

We choose again the **AAB** lattice with $\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} = 1$, and the exotic O6-plane is within one of the orbits $\{\Omega\mathcal{R}, \Omega\mathcal{R}\mathbb{Z}_2^{(1)}, \Omega\mathcal{R}\mathbb{Z}_2^{(2)}\}$. Imposing the supersymmetry condition (116) leads to the untwisted RR tadpole cancellation condition

$$\sum_a N_a X_a = 8 \tag{124}$$

on the **AAB** background with discrete torsion. The twisted RR tadpole conditions can be solved by adding four D6-branes a_m with different $\mathbb{Z}_2^{(i)}$ eigenvalues such that the sum wraps the bulk cycle a in table 42,

$$\begin{aligned}
\Pi_{a_m} =& \frac{1}{4}(\rho_1 + \rho_2) + \frac{(-1)^{\tau_{0,m}^{(1)}}}{4}\left(\varepsilon_1^{(1)} - \tilde{\varepsilon}_1^{(1)} + \varepsilon_2^{(1)} - \tilde{\varepsilon}_2^{(1)} + \varepsilon_3^{(1)} - \tilde{\varepsilon}_3^{(1)}\right) \\
&+ \frac{(-1)^{\tau_{0,m}^{(2)}}}{4}\left(\tilde{\varepsilon}_1^{(2)} - \varepsilon_2^{(2)} + \tilde{\varepsilon}_4^{(2)}\right) \\
&+ \frac{(-1)^{\tau_{0,m}^{(3)}}}{4}\left(-2\,\varepsilon_1^{(3)} + \tilde{\varepsilon}_1^{(3)} + \varepsilon_2^{(3)} - 2\,\tilde{\varepsilon}_2^{(3)} - 2\,\varepsilon_4^{(3)} + \tilde{\varepsilon}_4^{(3)}\right)
\end{aligned} \tag{125}$$

where all discrete displacements and Wilson lines have been set to zero, $\vec{\sigma} = \vec{\tau} = 0$, and the exceptional contributions are read off from table 55. Since $\sum_m N_{a_m}\Pi_{a_m} = 2\,(\rho_1 + \rho_2)$, the K-theory contraint is trivially fulfilled.

We use the following assignment of $\mathbb{Z}_2^{(i)}$ eigenvalues to the four fractional D6-branes.

brane	$\tau_{0,m}^{(1)}$	$\tau_{0,m}^{(2)}$	$\tau_{0,m}^{(3)}$
a_0	0	0	0
a_1	0	1	1
a_2	1	0	1
a_3	1	1	0

The orientifold image branes are computed using tables 34 to 36 with the result

$$\begin{aligned}
\Pi'_{a_m} =& \frac{1}{4}(\rho_1 + \rho_2) + \eta_{(1)}\frac{(-1)^{\tau_{0,m}^{(1)}}}{4}\left(-\tilde{\varepsilon}_1^{(1)} + \varepsilon_2^{(1)} - \tilde{\varepsilon}_4^{(1)}\right) \\
&+ \eta_{(2)}\frac{(-1)^{\tau_{0,m}^{(2)}}}{4}\left(-\varepsilon_1^{(2)} + \tilde{\varepsilon}_1^{(2)} - \varepsilon_2^{(2)} + \tilde{\varepsilon}_2^{(2)} - \varepsilon_3^{(2)} + \tilde{\varepsilon}_3^{(2)}\right) \\
&+ \eta_{(3)}\frac{(-1)^{\tau_{0,m}^{(3)}}}{4}\left(-\,\varepsilon_1^{(3)} + 2\,\tilde{\varepsilon}_1^{(3)} + 2\,\varepsilon_2^{(3)} - \tilde{\varepsilon}_2^{(3)} - \varepsilon_5^{(3)} + 2\,\tilde{\varepsilon}_5^{(3)}\right),
\end{aligned} \tag{126}$$

which leads to vanishing intersection numbers

$$\Pi_{a_m} \circ \Pi_{a_n} = \Pi_{a_m} \circ \Pi'_{a_n} = 0. \tag{127}$$

One can check explicitly that the terms vanish for each untwisted and twisted sector separately. The massless spectrum is thus completely non-chiral.

The gauge group is $U(2)^4$, and the massless matter spectrum splits into a part which is independent of the choice of the exotic O6-plane

- bifundamental representations on parallel D6-branes $a_m a_n$ with $m \neq n$

  $$\left[(\mathbf{2},\overline{\mathbf{2}},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\overline{\mathbf{2}}) + (\mathbf{2},\mathbf{1},\overline{\mathbf{2}},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\overline{\mathbf{2}}) + (\mathbf{2},\mathbf{1},\mathbf{1},\overline{\mathbf{2}}) + (\mathbf{1},\mathbf{2},\overline{\mathbf{2}},\mathbf{1}) + c.c.\right]$$

- bifundamental representations at intersections $a_m(\omega^k a_n)_{k=1,2}$,

  $$2 \times \left[(\mathbf{2},\overline{\mathbf{2}},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\overline{\mathbf{2}}) + c.c.\right] + \left[(\mathbf{2},\mathbf{1},\overline{\mathbf{2}},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\overline{\mathbf{2}}) + c.c.\right],$$

  which stem from one intersection point that is $\mathbb{Z}_2^{(1)}$ and $\mathbb{Z}_2^{(3)}$ invariant plus a pair of $\mathbb{Z}_2^{(3)}$ invariant intersections, which are exchanged under $\mathbb{Z}_2^{(1)}$,

and a part which depends on the choice of the exotic O6-plane and contains:

- bifundamental representations at intersections $a_m(\theta^k a'_m)_{k=0,1,2}$

  $$\begin{aligned} &\frac{7+\eta_{(1)}-\eta_{(2)}-\eta_{(3)}}{4}[(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + c.c.], \\ +&\frac{7-\eta_{(1)}+\eta_{(2)}-\eta_{(3)}}{4}[(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) + c.c.], \\ +&\frac{7-\eta_{(1)}-\eta_{(2)}+\eta_{(3)}}{4}[(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + c.c.], \end{aligned}$$

- two non-chiral pairs of antisymmetric representations per $U(2)$ gauge factor if $\eta_{\Omega\mathcal{R}} = \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(3)}} = 1$ or one pair of symmetric representations if $\eta_{\Omega\mathcal{R}} = -1$.

The diagonal Abelian factor $U(1) = \sum_{m=0}^{3} U(1)_m$ stays massless, while the other three linear combinations are anomalous and receive a mass by the generalised Green-Schwarz mechanism. The K-theory constraint is trivially fulfilled for $N_{a_m} = 2$ for all stacks.

Note again, that there are no multiplets in the adjoint representation. The D6-branes are thus again completely rigid.

7 Conclusions and Outlook

In this work, we gave the complete list of supersymmetric D6-brane model building rules on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ orientifolds with discrete torsion and factorisable tori.

For $2M = 2$, in section 3 we completed the classification of orientifold invariant D6-branes in the presence of discrete torsion, which can lead to new K-theory constraints for backgrounds with at least one tilted torus, which has to our knowledge not been noted before. We also found that the computation of intersection numbers is even on this orbifold not sufficient in order to derive the full matter spectrum, if D6-branes are parallel on some two-torus.

For $2M = 4$, we found that for both cases without and with discrete torsion, D6-branes wrap the same half-bulk three-cycles, and in section 4.2 we showed that three generation left-right symmetric models or Standard Model vacua with all quarks realised as bifundamental representations are excluded, and $SU(5)$ GUT vacua are by the same simple arguments excluded on four of the six inequivalent background lattices. We expect that a more detailed case-by-case study, which goes beyond the scope of this paper, rules also out the remaining model building possibilities of some right-handed quarks realised as antisymmetric representations of the QCD stack.

For $2M = 6, 6'$, we constructed the complete lattice of untwisted and $\mathbb{Z}_2$ twisted three-cycles, on which D6-brane model building can be performed. We gave a chiral example on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ without and with discrete torsion in section 5.5 and 5.6, respectively. The D6-branes in the latter case were completely rigid. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ without discrete torsion, supersymmetry excludes chirality due to $b_3^{\text{no torsion}} = 2$, but with discrete torsion, there is ample possiblity for chiral spectra. In section 6.5, we showed for an example that completely rigid D6-branes also occur on this orbifold background.

The $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ orientifolds with discrete torsion look very promising in view of obtaining three Standard Model generations. However, rigid D6-branes without any adjoint representation arising at intersections of orbifold image branes only occur under special circumstances, namely on the former orbifold rigidity constrains the wrapping numbers on $T^2_{(2)} \times T^2_{(3)}$ to just 3×3 possibilities, while also requiring the relation $\sigma_2\tau_2 = \sigma_3\tau_3$ for the discrete displacements and Wilson lines, as detailed in appendix B.2.1. For the latter orbifold, the condition on rigid D6-branes is not as straightforwardly classified, but still very restrictive. This considerably narrows the search for phenomenologically interesting models, since the cases of completely rigid D6-branes are of paramount interest as candidates for the QCD and electro-weak stacks. We will come back to analysing model building on these two orbifold backgrounds in future work [87]. If Standard Model vacua

on rigid D6-branes are found, we will be in the unique position of being able to compute the low-energy effective field theory exactly by means of conformal field theory - in contrast to the present focus on F-theory models on smooth Calabi-Yau manifolds, for which there is at present no such tool available. For the expected Standard Model vacua on rigid D6-branes, it will also be of great interest to investigate the blow-up of fixed points and if twisted moduli are stabilised at the orbifold fixed point, but also if de Sitter vacua are feasible, see e.g. the analogous discussion on heterotic orbifolds [88, 89]. The blown-up models will also provide a more geometric understanding of our constructions as discussed e.g. in [**?**]. By using M/F-theory duality, the blown-up IIA orbifold results will help to derive the missing low-energy properties of F-theory models.

As a next step it will also be interesting to introduce closed string background fluxes in the set-up, in particular as means to stabilise the untwisted Kähler moduli and the dilaton.

Finally, Standard Models on rigid D6-branes in orbifold backgrounds with torsion might provide explicit examples for the proposed phenomenon of low-scale string signatures at the LHC, see e.g. [90–93].

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A The IIA closed string spectrum on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ without and with discrete torsion

A.1 The massless closed IIA string spectrum on Calabi-Yau manifolds and their orientifolds

The massless closed string spectrum of IIA string theory on smooth Calabi-Yau-threefolds and its orientifold by $\Omega\mathcal{R}$ has been discussed in detail in [94]. The multiplicities of multiplets in terms of Hodge numbers and the bosonic content of each multiplet are summarised in table 43, where the four dimensional bosonic degrees of freedom are obtained from the

Four dimensionsal closed IIA spectra on Calabi-Yaus and their orientifolds					
$\mathcal{N}=2$ multiplet	mult.	bosons	$\mathcal{N}=1$ multiplet	mult.	bosons
gravity	1	$(G_{\mu\nu}, A^0)$	gravity	1	$(G_{\mu\nu})$
tensor	1	$(B_{\mu\nu}, \phi, \zeta^0, \tilde{\zeta}_0)$	linear (dilaton-axion)	1	(ϕ, ξ^0)
hyper	h_{21}	$(z^k, \bar{z}_k, \zeta^k, \tilde{\zeta}_k)$	chiral (complex structures)	h_{21}	(c^k, ξ^k)
vector	h_{11}	(A^i, v^i, b^i)	chiral (Kähler moduli)	h_{11}^-	(v^i, b^i)
			vector	h_{11}^+	(A^i)

Table 43: The massless $\mathcal{N}=2$ and $\mathcal{N}=1$ four dimensional multiplets for type IIA string theory on a Calabi-Yau manifold (left) and its orientifold (right). The vector, hyper and chiral multiplets are counted by the Hodge numbers of the manifold and their transformation under $\Omega\mathcal{R}$. The bosonic field content is listed explicitly.

ten dimensional RR-fields $C_{\rm odd}$, the NSNS-two form B_2 and the Kähler form $J_{\rm Kähler}$ (which is a reparameterisation of the metric G) of the Calabi-Yau manifold via integrals over two- and three-cycles as follows,

$$\begin{aligned}
&A^0 = C_1,\\
&A^i = \int_{i^{th}\ (1,1)-\text{cycle}} C_3, \quad v^i = \int_{i^{th}\ (1,1)-\text{cycle}} J_{\text{Kähler}}, \quad b^i = \int_{i^{th}\ (1,1)-\text{cycle}} B_2, \quad i = 1\ldots h_{11},\\
&\zeta^K = \int_{K^{th}\ 3\text{-cycle}} C_3, \qquad \tilde{\zeta}_K = \int_{K^{th}\ \text{dual 3-cycle}} C_3, \qquad k = 0\ldots h_{21},\\
&z^k, \bar{z}_k \quad \text{complex structure deformations} \qquad\qquad k = 1\ldots h_{21}.
\end{aligned} \tag{128}$$

Making use of the worldsheet parity of the ten dimensional massless IIA fields,

$$\Omega(G) = G, \qquad \Omega(\phi) = \phi, \qquad \Omega(B_2) = -B_2, \qquad \Omega(C_1) = -C_1, \qquad \Omega(C_3) = C_3, \quad (129)$$

leads to the $\mathcal{N} = 1$ fields in four dimensions,

$$\begin{aligned}
&A^i = \int_{i^{th}\,\mathcal{R}\text{even (1,1)-cycle}} C_3, \qquad v^i = \int_{i^{th}\,\mathcal{R}\text{odd (1,1)-cycle}} J_{\text{Kähler}}, \qquad b^i = \int_{i^{th}\,\mathcal{R}\text{odd (1,1)-cycle}} B_2, \\
&\xi^K = \int_{K^{th}\mathcal{R}\text{even 3-cycle}} C_3, \qquad k = 1 \ldots h_{21}, \\
&c^k \sim z^k + \bar{z}_k \qquad \text{complex structure deformations} \qquad k = 1 \ldots h_{21}.
\end{aligned} \tag{130}$$

In the orbifold limit $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$, both without and with discrete torsion, the closed string spectrum can be either computed by the same means using all two- and three-cycles from the bulk and twisted sectors. Alternatively, the closed string spectrum can be obtained at the orbifold point in a separate computation (see the following section), and the decomposition of the Hodge number h_{11} into $\mathcal{R}$ even and odd cycles per twist sector can be read off from the number of chiral multiplets. This information is important for the issue of moduli stabilisation, which goes beyond the scope of the present work.

A.2 The massless closed IIA string spectrum on orbifolds

The untwisted closed string spectrum can be computed as follows: any left-moving string is given by the four-vector $|s_0, \vec{s}\rangle$, and right moving strings are parameterised by $|s'_0, \vec{s}'\rangle$, where the first entry encodes the spin along the four non-compact dimensions and the remaining three entries spins per complex two-torus.

In the NS-sectors, all entries $s_i, s'_i \in \mathbb{Z}$ with $|0, \vec{0}\rangle_{\text{NS}}$ the tachyonic vacuum state. The GSO projection selects physical states with both $\sum_i s_i$ and $\sum_i s'_i$ odd. The massless untwisted left-moving NS-sector states are $|\pm 1, \vec{0}\rangle \equiv \psi^{\mu}_{-1/2}|0\rangle_{\text{NS}}$ and $|0, \underline{\pm 1_i, 0_j, 0_k}\rangle \equiv \psi^{i}_{-1/2}|0\rangle_{\text{NS}}, \psi^{\bar{i}}_{-1/2}|0\rangle_{\text{NS}}$ and similarly for the right-moving NS-sector with oscillators $\tilde{\psi}$.

The R sectors have half-integer entries s_i, s'_i with the GSO projection enforcing $\sum_i s_i$ odd and $\sum_i s'_i$ even for the IIA string theory. The massless R sector states in the left moving sector are $|\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle_{\text{R}} \equiv |-+++\rangle_{\text{R}}$ or any state with an even number of signs flip, whereas the massless right-moving R states are of the form $|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle_{\text{R}} \equiv |++++\rangle_{\text{R}}$ or an even number of signs flipped.

The orbifold projectors act identically on the left- and right-moving sectors by a phase,

$$\theta : |s_0, \vec{s}\rangle \longrightarrow e^{2\pi i \vec{v}\cdot\vec{s}}|s_0, \vec{s}\rangle \qquad \text{and} \qquad \omega : |s_0, \vec{s}\rangle \longrightarrow e^{2\pi i \vec{w}\cdot\vec{s}}|s_0, \vec{s}\rangle, \tag{131}$$

whereas the orientifold projection flips the spins along the compact directions while exchanging left- and right-moving sectors,

$$\begin{aligned} \Omega\mathcal{R} : |s_0, \vec{s}\rangle_{\rm NS}|s_0', \vec{s}'\rangle_{\rm NS} &\longrightarrow |s_0', -\vec{s}'\rangle_{\rm NS}|s_0, -\vec{s}\rangle_{\rm NS}, \\ \Omega\mathcal{R} : |s_0, \vec{s}\rangle_{\rm R}|s_0', \vec{s}'\rangle_{\rm R} &\longrightarrow -|s_0', -\vec{s}'\rangle_{\rm R}|s_0, -\vec{s}\rangle_{\rm R}. \end{aligned} \tag{132}$$

In the RR-sector, the exchange of the left- and right-moving sector is accompanied by a minus sign. As a result, the untwisted IIA closed string spectra before and after the orientifold projection are given in table 44.

The untwisted closed string spectrum is independent of the choice of discrete torsion.

The twisted $\mathcal{N}=2$ spectrum depends on the choice of discrete torsion, and moreover the orientifold projected twisted $\mathcal{N}=1$ states depend on the choice of the exotic O6-plane as follows. Closed string states in the $n\vec{v}+m\vec{w}$ twisted sector are obtained from the states

$$|p_0, \vec{p}\rangle \equiv |s_0, \vec{s} + (n\vec{v}+m\vec{w})\rangle, \qquad |p_0', \vec{p}'\rangle \equiv |s_0', \vec{s}' - (n\vec{v}+m\vec{w})\rangle, \tag{133}$$

which shows that the $\Omega\mathcal{R}$ projection preserves each twist sector. The GSO projection is the same as in the untwisted sector, but the orbifold action in the case of discrete torsion is modified by the signs discussed in section 2.1.2.

The masses of the twisted states are computed from

$$\frac{\alpha'}{4}m^2 = \frac{1}{2}p^2 + E_T - \frac{1}{2} \qquad \text{with} \qquad E_T = \frac{1}{2}\sum_{i=1}^{3}|nv_i + mw_i|\left(1 - |nv_i + mw_i|\right) \tag{134}$$

where $|nv_i + mw_i| \in [0,1)$ (up to an integer shift which might need to be performed).

The complete twisted closed string spectrum for the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds without and with discrete torsion is displayed in table 45.

Under the orientifold projection, the $\theta^k\omega^l$ twisted closed string sector picks up a sign $\eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\theta^k\omega^l}$ as explained in section 2.1.2. Since $\eta_{\Omega\mathcal{R}\theta^k\omega^l} = \eta_{\Omega\mathcal{R}\theta^k\omega^{l+2}} = \eta_{\Omega\mathcal{R}\theta^{k+2}\omega^l} = \eta_{\Omega\mathcal{R}\theta^{k+2}\omega^{l+2}}$, only three different non-trivial prefactors arise which we labelled for $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}_6'$ by $\eta_{(i)} \equiv \eta_{\Omega\mathcal{R}}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}$. As a result, the number of Kähler moduli and Abelian vectors does not only depend on the choice of orbifold and discrete torsion, but also on the choice of the exotic O6-plane, as can be seen in the complete list of the orientifolded twisted closed string spectra in table 46.

Massless open string states can be counted by e.g. taking the expressions (133) for the right-moving NS and R sectors and replacing the twist sector $n\vec{v}+m\vec{w}$ by the relative angles $\vec{\varphi}$ of the D6-branes on which the open string under question ends.

Bosonic part of the untwisted massless closed string spectrum on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ orbifolds and orientifolds			
$T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$		$T^6/(\mathbb{Z}_2\times\mathbb{Z}_{2M}\times\Omega\mathcal{R})$	
particle	massless state	particle	massless state
NS-NS sector (i) universal			
$G_{\mu\nu}+B_{\mu\nu}+\phi$	$\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}\vert 0\rangle_{\rm NSNS}$	$G_{\mu\nu}+\phi$	$\left(\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}+\psi^{\nu}_{-1/2}\tilde{\psi}^{\mu}_{-1/2}\right)\vert 0\rangle_{\rm NSNS}$
$(v^i,b^i)_{i=1,2,3}$	$\psi^{i}_{-1/2}\tilde{\psi}^{\bar{i}}_{-1/2}\vert 0\rangle_{\rm NSNS},\quad \psi^{\bar{i}}_{-1/2}\tilde{\psi}^{i}_{-1/2}\vert 0\rangle_{\rm NSNS}$	$(v^i,b^i)_{i=1,2,3}$	$\psi^{i}_{-1/2}\tilde{\psi}^{\bar{i}}_{-1/2}\vert 0\rangle_{\rm NSNS},\quad \psi^{\bar{i}}_{-1/2}\tilde{\psi}^{i}_{-1/2}\vert 0\rangle_{\rm NSNS}$
NS-NS sector (ii) $\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_6$ only			
$z^1,\bar{z}_1$	$\psi^{1}_{-1/2}\tilde{\psi}^{1}_{-1/2}\vert 0\rangle_{\rm NSNS},\quad \psi^{\bar{1}}_{-1/2}\tilde{\psi}^{\bar{1}}_{-1/2}\vert 0\rangle_{\rm NSNS}$	c^1	$\left(\psi^{1}_{-1/2}\tilde{\psi}^{1}_{-1/2}+\psi^{\bar{1}}_{-1/2}\tilde{\psi}^{\bar{1}}_{-1/2}\right)\vert 0\rangle_{\rm NSNS}$
NS-NS sector (iii) $\mathbb{Z}_2\times\mathbb{Z}_2$ only			
$(z^i,\bar{z}_i)_{i=2,3}$	$\psi^{i}_{-1/2}\tilde{\psi}^{i}_{-1/2}\vert 0\rangle_{\rm NSNS},\quad \psi^{\bar{i}}_{-1/2}\tilde{\psi}^{\bar{i}}_{-1/2}\vert 0\rangle_{\rm NSNS}$	$(c^i)_{i=2,3}$	$\left(\psi^{i}_{-1/2}\tilde{\psi}^{i}_{-1/2}+\psi^{\bar{i}}_{-1/2}\tilde{\psi}^{\bar{i}}_{-1/2}\right)\vert 0\rangle_{\rm NSNS}$
RR sector (i) universal			
$(\zeta^0,\tilde{\zeta}_0)$	$\left\{\begin{array}{l}\vert -+++\rangle\vert ++++\rangle_{\rm RR}\\ \vert +---\rangle\vert ----\rangle_{\rm RR}\end{array}\right.$	ξ^0	$\vert -+++\rangle\vert ++++\rangle_{\rm RR}-\vert +---\rangle\vert ----\rangle_{\rm RR}$
$(A^0,A^i)_{i=1,2,3}$	$\left\{\begin{array}{l}\vert -+++\rangle\vert ----\rangle_{\rm RR}\\ \vert -+--\rangle\vert --++\rangle_{\rm RR}\\ \vert ---+\rangle\vert -++-\rangle_{\rm RR}\\ \vert --+-\rangle\vert -+-+\rangle_{\rm RR}\end{array}\right.,\left\{\begin{array}{l}\vert +-++\rangle\vert ++--\rangle_{\rm RR}\\ \vert ++-+\rangle\vert +-+-\rangle_{\rm RR}\\ \vert +++-\rangle\vert +--+\rangle_{\rm RR}\\ \vert +---\rangle\vert ++++\rangle_{\rm RR}\end{array}\right.$	$\emptyset$	
RR sector (ii) $\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_4,\mathbb{Z}_2\times\mathbb{Z}_6$ only			
$(\zeta^1,\tilde{\zeta}_1)$	$\left\{\begin{array}{l}\vert +-++\rangle\vert --++\rangle_{\rm RR}\\ \vert -+--\rangle\vert ++--\rangle_{\rm RR}\end{array}\right.$	ξ^1	$\vert +-++\rangle\vert --++\rangle_{\rm RR}-\vert -+--\rangle\vert ++--\rangle_{\rm RR}$
RR sector (iii) $\mathbb{Z}_2\times\mathbb{Z}_2$ only			
$(\zeta^i,\tilde{\zeta}_i)_{i=2,3}$	$\left\{\begin{array}{l}\vert ---+\rangle\vert +--+\rangle_{\rm RR}\\ \vert --+-\rangle\vert +-+-\rangle_{\rm RR}\\ \vert ++-+\rangle\vert -+-+\rangle_{\rm RR}\\ \vert +++-\rangle\vert -++-\rangle_{\rm RR}\end{array}\right.$	$(\xi^i)_{i=2,3}$	$\left\{\begin{array}{l}\vert ---+\rangle\vert +--+\rangle_{\rm RR}-\vert +++-\rangle\vert -++-\rangle_{\rm RR}\\ \vert --+-\rangle\vert +-+-\rangle_{\rm RR}-\vert ++-+\rangle\vert -+-+\rangle_{\rm RR}\end{array}\right.$
$(h^+_{11},h^-_{11})^U=(0,3)$ $\quad h^U_{21}=3(\mathbb{Z}_2\times\mathbb{Z}_2),\ 1(\mathbb{Z}_2\times\mathbb{Z}_4$ and $\mathbb{Z}_2\times\mathbb{Z}_6),\ 0(\mathbb{Z}_2\times\mathbb{Z}_6')$			

Table 44: The massless untwisted $\mathcal{N}=2$ and $\mathcal{N}=1$ four dimensional multiplets for type IIA string theory on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ (left) and its orientifold by $\Omega\mathcal{R}$ (right). Only bosonic states are listed; the fermionic superpartners arise in the NS-R and R-NS sectors. The untwisted spectrum is independent of the choice of discrete torsion and the exotic O6-plane.

B The open string spectrum on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ via Chan-Paton labels and gauge threshold amplitudes

B.1 Chan-Paton labels

The Chan-Paton label λ associated to some open string state transforms as follows under the $\mathbb{Z}_2$ orbifold projections,

$$\mathbb{Z}_2^{(i)}: \quad \lambda|\text{state}\rangle \longrightarrow c_{\text{state}}^{\mathbb{Z}_2^{(i)}}\left(\gamma_{\mathbb{Z}_2^{(i)}}\lambda\gamma^{-1}_{\mathbb{Z}_2^{(i)}}\right)|\text{state}\rangle. \tag{135}$$

In the absence of discrete Wilson lines, $c^{\mathbb{Z}_2^{(i)}}_{\text{state}} = \pm 1$ is simply the $\mathbb{Z}_2^{(i)}$ eigenvalue of the massless state. Orbifold generators other than $\mathbb{Z}_2$ change the positions of the D6-branes where the open string ends on, thereby identifying orbifold images.

In order to evaluate the Chan-Paton labels, representations of the $\mathbb{Z}_2$ gamma matrices are needed. We start with

$$\gamma_{\mathbb{Z}_2^{(1)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_{\mathbb{Z}_2^{(2)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma_{\mathbb{Z}_2^{(3)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{136}$$

and check the consistency conditions: $\gamma^2_{\mathbb{Z}_2^{(i)}} = \mathbf{1}$ for all i and $\gamma_{\mathbb{Z}_2^{(1)}} \cdot \gamma_{\mathbb{Z}_2^{(2)}} = \gamma_{\mathbb{Z}_2^{(3)}}$. A Chan-Paton label contains the representations

$$\lambda = \begin{pmatrix} (\mathbf{N}^1_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^2_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^3_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^4_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^4_b) \end{pmatrix} \tag{137}$$

of $\prod_{i=1}^4 U(N_a^i) \times U(N_b^i)$. A single $\mathbb{Z}_2^{(k)}$ projection acts as follows.

$(\mathbb{Z}_2^{(1)}, \mathbb{Z}_2^{(2)}, \mathbb{Z}_2^{(3)})$	λ	$(\mathbb{Z}_2^{(1)}, \mathbb{Z}_2^{(2)}, \mathbb{Z}_2^{(3)})$	λ
$(+,*,*)$	$\begin{pmatrix} (\mathbf{N}^1_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^2_b) & 0 & 0 \\ (\mathbf{N}^2_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^2_b) & 0 & 0 \\ 0 & 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^4_b) \\ 0 & 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^4_b) \end{pmatrix}$	$(-,*,*)$	$\begin{pmatrix} 0 & 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^4_b) \\ 0 & 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^3_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^3_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^2_b) & 0 & \\ (\mathbf{N}^4_a, \overline{\mathbf{N}}^1_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^2_b) & 0 & 0 \end{pmatrix}$
$(*,+,*)$	$\begin{pmatrix} (\mathbf{N}^1_a, \overline{\mathbf{N}}^1_b) & 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^3_b) & 0 \\ 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^2_b) & 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^3_a, \overline{\mathbf{N}}^1_b) & 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^3_b) & 0 \\ 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^2_b) & 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^4_b) \end{pmatrix}$	$(*,-,*)$	$\begin{pmatrix} 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^2_b) & 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^2_a, \overline{\mathbf{N}}^1_b) & 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^3_b) & 0 \\ 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^2_b) & 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^4_a, \overline{\mathbf{N}}^1_b) & 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^3_b) & 0 \end{pmatrix}$
$(*,*,+)$	$\begin{pmatrix} (\mathbf{N}^1_a, \overline{\mathbf{N}}^1_b) & 0 & 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^4_b) \\ 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^2_a, \overline{\mathbf{N}}^3_b) & 0 \\ 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^3_a, \overline{\mathbf{N}}^3_b) & 0 \\ (\mathbf{N}^4_a, \overline{\mathbf{N}}^1_b) & 0 & 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^4_b) \end{pmatrix}$	$(*,*,-)$	$\begin{pmatrix} 0 & (\mathbf{N}^1_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^1_a, \overline{\mathbf{N}}^3_b) & 0 \\ (\mathbf{N}^2_a, \overline{\mathbf{N}}^1_b) & 0 & 0 & (\mathbf{N}^2_a, \overline{\mathbf{N}}^4_b) \\ (\mathbf{N}^3_a, \overline{\mathbf{N}}^1_b) & 0 & 0 & (\mathbf{N}^3_a, \overline{\mathbf{N}}^4_b) \\ 0 & (\mathbf{N}^4_a, \overline{\mathbf{N}}^2_b) & (\mathbf{N}^4_a, \overline{\mathbf{N}}^3_b) & 0 \end{pmatrix}$

If some λ is subject to two $\mathbb{Z}_2^{(i)}$ projections, the third one is automatically fulfilled.

The open string states can be computed analogously to the right-moving sector of the closed string states. This leads to the massless NS and R states with given $\mathbb{Z}_2^{(i)}$ eigenvalues in table 47. In the absence of discrete Wilson lines, parallel D6-branes thus provide three non-chiral $\mathcal{N} = 2$ hyper multiplets with different $\mathbb{Z}_2^{(i)}$ eigenvalues. D6-branes at angle $\pi(0, \phi, -\phi)$ provide two chiral multiplets of opposite chirality and with different $\mathbb{Z}_2^{(2)}$ and

$\mathbb{Z}_2^{(3)}$ eigenvalues. If the D6-brane intersection point does not coincide with a $\mathbb{Z}_2^{(1)}$ fixed point, a non-chiral $\mathcal{N}=2$ hyper multiplet arises, which is identified with another hyper multiplet at the $\mathbb{Z}_2^{(1)}$ image of the intersection point.

This means that not even for the so far existing literature on $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ [33, 74], the open string spectrum is fully determined by intersection numbers! The running of gauge couplings can only be computed if these non-chiral states are included in the beta function coefficient.

B.2 The gauge thresholds and beta functions

The complete massless charged matter content can be determined if the corresponding beta function coefficient is known. The beta function coefficient arises in the computation of threshold corrections to the gauge kinetic function as follows: these threshold corrections can be derived by using a gauged partition function to describe the Annulus and Möbius strip amplitudes and expanding in powers of the newly introduced non-compact magnetic field, see e.g. [23] and references therein. As a result, one obtains a sum of three terms: (1) a tadpole, which cancels among all possible contributions from various D-branes in a RR tadpole free configuration; (2) a term proportional to $1/\varepsilon$, where ε is the power of the dimensional regularisation and $1/\varepsilon$ is identified with $\ln\frac{M^2_{\rm string}}{\mu^2}$, whose numerical prefactor is the beta function contribution from the D6-brane and O6-plane configuration under consideration; (3) a finite term, the actual gauge threshold due to massive strings.

We can thus use the extensive inspection of gauge thresholds on $T^6/\mathbb{Z}_{2N}$ performed in [79] to find beta function coefficients on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ without and with discrete torsion and compare with the field theoretic expression for the beta function coefficients

$$\begin{aligned}
b_{SU(N_a)} &= -\,3\,N_a+\sum_{b\neq a}\frac{N_b}{2}\left(\varphi^{ab}+\varphi^{ab'}\right)+N_a\,\varphi^{\mathbf{Adj}_a}+\frac{N_a}{2}\,\left(\varphi^{\mathbf{Sym}_a}+\varphi^{\mathbf{Anti}_a}\right)\\
&\quad+\left(\varphi^{\mathbf{Sym}_a}-\varphi^{\mathbf{Anti}_a}\right),\\
\left.\begin{array}{c} b_{Sp(2M_x)} \\ b_{SO(2M_x)}\end{array}\right\} &= -\,3\,(M_x\pm 1)+\sum_{a\neq x}\frac{N_a}{2}\varphi^{ax}+M_x\,\left(\varphi^{\mathbf{Sym}_x}+\varphi^{\mathbf{Anti}_x}\right)\\
&\quad+\left(\varphi^{\mathbf{Sym}_x}-\varphi^{\mathbf{Anti}_x}\right).
\end{aligned}\tag{138}$$

The advantage of this method of counting the massless spectrum is its completeness, whereas it is blind to chiralities. The derivation of the beta function coefficients can therefore not replace the computation of the chiral spectrum via three-cycle intersection numbers.

B.2.1 Rigid D6-branes without adjoint matter

The contributions to beta function coefficients from bifundamental and adjoint representations of $SU(N_a)$ are displayed in table 48. Using the expressions (59), (82) and (113) for the torus wrapping numbers of orbifold images on $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$, $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$ and $T^6/\mathbb{Z}_2\times\mathbb{Z}_6'$, respectively, one obtains

$$
\begin{array}{ll}
T^6/\mathbb{Z}_2\times\mathbb{Z}_4: & I^{(2\cdot3)}_{a(\theta a)} = -\prod_{i=2}^3\left((n_a^i)^2+(m_a^i)^2\right), \\
T^6/\mathbb{Z}_2\times\mathbb{Z}_6: & I^{(2\cdot3)}_{a(\theta a)} = I^{(2\cdot3)}_{a(\theta^2 a)} = -\prod_{i=2}^3\left((n_a^i)^2+n_a^i m_a^i+(m_a^i)^2\right), \\
T^6/\mathbb{Z}_2\times\mathbb{Z}_6': & I_{a(\theta a)} = -I_{a(\theta^2 a)} = -\prod_{i=1}^3\left((n_a^i)^2+n_a^i m_a^i+(m_a^i)^2\right).
\end{array}
$$

Completely rigid D6-branes with no adjoint representation arising at orbifold images is only possible if $|I^{(2\cdot3)}_{(\theta a)}| = I^{\mathbb{Z}_2^{(1)},(2\cdot3)}_{(\theta a)}$ for $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$ with discrete torsion, whereas it is impossible for $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$. For $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$ one has $n^2+nm+m^2=\frac{3}{4}(n+m)^2+\frac{1}{4}(n-m)^2$, which only has solutions to no adjoint representation for $(n,m)\in\{(1,0),(0,1),(1,-1)\}$, all with $n^2+nm+m^2=1$. The necessary conditions on the absence of adjoint representations needs to be supplemented by the condition that $I^{\mathbb{Z}_2^{(1)},(2\cdot3)}_{(\theta a)}>0$, which restricts the combinations of discrete Wilson lines and displacements on $T^2_{(2)}\times T^2_{(3)}$ due to the following relations

$$
\begin{array}{ccc}
i\in\{2,3\} & (n_a^i,m_a^i) & I^{\mathbb{Z}_2^{(1)},(i)}_{(\theta a)} \\
\mathbb{Z}_2\times\mathbb{Z}_6 & (1,0),(0,1),(1,-1) & (-1)^{\sigma_i^a\tau_i^a}.
\end{array}
$$

This has been derived by inspection of the intersection points along the lines of appendix A.1 in [79]. For $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$, supersymmetric solutions for $SU(N_a)$ gauge factors without any adjoint representation are found by choosing an appropriate one-cycle on $T^2_{(1)}$. For $T^6/\mathbb{Z}_2\times\mathbb{Z}_6'$, the situation is different since the torus intersection number can be cancelled by the sum of all three $\mathbb{Z}_2^{(i)}$ invariant intersection numbers, see the last row in table 48. In the example in section 6.5, the cancellation occured due to $I^{\mathbb{Z}_2^{(1)}}_{a_m(\omega^k a_m)} = -I^{\mathbb{Z}_2^{(2)}}_{a_m(\omega^k a_m)}$ and $I^{\mathbb{Z}_2^{(3)}}_{a_m(\omega^k a_m)} = -I_{a_m(\omega^k a_m)}$, which shows the existence of completely rigid D6-branes on this orbifold background.

The condition of completely rigid D6-branes severly resctrict the search for fully-fledged Standard Model vacua, which we will address in future work [87].

B.2.2 $Sp(2M)$ and $SO(2M)$ gauge factors

The $\Omega\mathcal{R}$ invariant three-cycles can be easily classified, as done in sections 3.3, 4.3, 5.3 and 6.3 for the $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ backgrounds without and with discrete torsion for $2M=2,4,6$

and $6'$, respectively. A priori, the gauge group which D6-branes wrapped on these cycles support, can be either of $SO(2N)$ or $Sp(2N)$ type.

One way of determining the correct group assignment consists of extracting the $\Omega\mathcal{R}$ eigenvalue of the massless open string state from the Möbius strip amplitude and finding a viable $\Omega\mathcal{R}$ projection matrix for the Chan-Paton label of a given D6-brane configuration.

However, the method of extracting the beta function coefficient from the gauge threshold computation turns out to be more economic. As discussed in detail in [79], the expressions for $b_{SO(2N)}$ and $b_{Sp(2N)}$ in terms of intersection numbers are (up to a global factor of $\frac{1}{2}$) the same as for $b_{SU(N)}$ in tables 48 and 49. From the first two rows of the latter, one can extract the type of gauge group supported by orientifold invariant D6-branes:

- since $\tilde{I}_x^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)},(j\cdot k)} = -4$ for any of the orientifold invariant three-cycles x on all $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds and all lattice orientations discussed in this article, the xx sector contribute $b_{SO/Sp(2N_x)} = -6$ in the case *without discrete torsion*, and therefore the gauge group is $Sp(2N_x)$ with three chiral multiplets in the antisymmetric representation;
- on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ *with discrete torsion*, the formulas for $b_{SU(N)}^{\text{no torsion}}$ in table 49 dressed with the charges $\eta_{\Omega\mathcal{R}}$ and $\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}$, lead to $SO(2N_x)$ gauge factors, if the stack of D6-branes is perpendicular to the orbit of exotic O6-planes, or $Sp(2N_x)$ otherwise.
- for $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ *with discrete torsion* and $2M \in \{2, 6, 6'\}$, the gauge group is $Sp(2N_x)$. For $b_i\sigma_i\tau_i = 0$, one can verify explicitly that this is due to the fact that orientifold invariant fractional D6-branes are parallel to the exotic O6-plane, cf. table 51.

Gauge groups of $SO(2N_x)$ type occur for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ with discrete torsion and $2M \in \{2, 6, 6'\}$ only in the presence of three exotic O6-planes, which is incompatible with supersymmetric D6-branes cancelling the untwisted RR tadpoles.

In general, there are more antisymmetric or symmetric representations of $Sp(2N_x)$ or $SO(2N_x)$ supported at intersections $x(\omega^k x)$, which need to be computed on a case by case basis.

C Tables for exceptional sectors in $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds with discrete torsion

In this appendix, we collect tables for the exceptional sectors of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ with $2M \in$

$\{2, 6, 6'\}$.

In table 50, the transformations of fixed points on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ for bulk parts parallel to some O6-plane are evaluated, which lead to the classification of orientifold invariant fractional D6-branes in table 51. The latter also holds for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$.

In tables 52 to 54, the assignment of exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ for a given choice of torus wrapping numbers (n^i, m^i), discrete displacements $\vec{\sigma}$ and Wilson lines τ is presented. In tables 55 and 56, the same relations for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ are given.

D From $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ orbifolds to $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$ without and with discrete torsion

The twisted sectors of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orbifolds without and with discrete torsion are inherited from various $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ sub-sectors, but the number of multiplets can be reduced by new identifications of fixed points. For example the sector twisted by $(0, \frac{1}{3}, -\frac{1}{3})$ contributes $h_{11} = 18$ on $T^4/\mathbb{Z}_3$, which splits into $(h_{11}, h_{21}) = (12, 6)$ on $T^6/\mathbb{Z}_6'$ which in turn is reduced to $(h_{11}, h_{21}) = (8, 2)$ on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ due to new $\mathbb{Z}_2$ identifications of the $\mathbb{Z}_3$ fixed points.

The Hodge numbers for factorisable $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ orbifolds are tabulated in table 57. Since the splitting of $h_{11} = h_{11}^+ + h_{11}^-$ has to our knowledge not been performed systematically before for orientifolds of $T^6/\mathbb{Z}_N$, we list the complete result in table 58.

In tables 59, 60, 61, 62, 63, we review how untwisted two- and three-cycles (denoted, e.g., by π_{35} and π_{135}) and twisted two- and three-cycles (two-cycles denoted by $d^{\mathbb{C}^2/\mathbb{Z}_N}$ for $N \neq 2$ and e for $\mathbb{Z}_2$ singularities, three-cycles denoted by ε or $\tilde{\varepsilon}$ if they stem from a $\mathbb{Z}_2$ twisted sector and δ or $\tilde{\delta}$ otherwise) of the $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds arise from $T^4/\mathbb{Z}_M$ and $T^6/\mathbb{Z}_N$ sectors. For completeness, we also include the case $T^6/\mathbb{Z}_6' = T^6/\mathbb{Z}_2 \times \mathbb{Z}_3$ which does not admit discrete torsion.

Twisted massless closed strings on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ without and with torsion

$T^6/$ V/H f.p. $(\#V,\#H)$	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	$\vec{v}$	$(\vec{v}+\vec{w})$	$(\vec{v}+2\vec{w})$	$(\vec{v}+3\vec{w})$
$\mathbb{Z}_2\times\mathbb{Z}_2$	$(0,\frac{1}{2},-\frac{1}{2})$			$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},0,-\frac{1}{2})$		
V	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle 00--\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert +-00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert -+00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$			$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle 0--0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert +00-\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert -00+\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle 0-0-\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert +0-0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert -0+0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$		
H	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00--\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert -+00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert +-00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$			$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0--0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert -00+\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert +00-\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0-0-\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert -0+0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert +0-0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$		
f.p.	$i,j\in\{1\ldots4\}$ on $T_2\times T_3$			$i,j\in\{1\ldots4\}$ on $T_1\times T_2$	$i,j\in\{1\ldots4\}$ on $T_1\times T_3$		
$(\#V,\#H)$	$(\frac{1+\eta}{2}\cdot16,\frac{1-\eta}{2}\cdot16)$			$(\frac{1+\eta}{2}\cdot16,\frac{1-\eta}{2}\cdot16)$	$(\frac{1+\eta}{2}\cdot16,\frac{1-\eta}{2}\cdot16)$		
$\mathbb{Z}_2\times\mathbb{Z}_4$	$(0,\frac{1}{4},-\frac{1}{4})$	$(0,\frac{1}{2},-\frac{1}{2})$		$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{4},-\frac{1}{4})$	$(\frac{1}{2},0,-\frac{1}{2})$	
V	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 00\frac{1}{4}\frac{3}{4}\rangle\lvert 00\frac{-1}{4}\frac{-3}{4}\rangle_{\rm NSNS}\\ \lvert 00\frac{-3}{4}\frac{-1}{4}\rangle\lvert 00\frac{3}{4}\frac{1}{4}\rangle_{\rm NSNS}\\ \lvert\frac{-1}{2}\frac{-1}{2}\frac{-1}{4}\frac{1}{4}\rangle\lvert\frac{-1}{2}\frac{1}{2}\frac{1}{4}\frac{-1}{4}\rangle_{\rm RR}\\ \lvert\frac{1}{2}\frac{1}{2}\frac{-1}{4}\frac{1}{4}\rangle\lvert\frac{1}{2}\frac{-1}{2}\frac{1}{4}\frac{-1}{4}\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\left\{\begin{matrix}\lvert 00++\rangle 00--\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert +-00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert -+00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$		$\left\{\begin{matrix}\lvert 0++0\rangle 0--0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert +00-\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert -00+\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0\frac{-1}{2}\frac{-1}{4}\frac{-1}{4}\rangle\lvert 0\frac{1}{2}\frac{1}{4}\frac{1}{4}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2}0\frac{1}{4}\frac{1}{4}\rangle\lvert\frac{1}{2}0\frac{-1}{4}\frac{-1}{4}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\left\{\begin{matrix}\lvert 0+0+\rangle 0-0-\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert +0-0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert -0+0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	
H	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 00\frac{1}{4}\frac{3}{4}\rangle\lvert 00\frac{3}{4}\frac{1}{4}\rangle_{\rm NSNS}\\ \lvert 00\frac{-3}{4}\frac{-1}{4}\rangle\lvert 00\frac{-1}{4}\frac{-3}{4}\rangle_{\rm NSNS}\\ \lvert\frac{-1}{2}\frac{-1}{2}\frac{-1}{4}\frac{1}{4}\rangle\lvert\frac{1}{2}\frac{-1}{2}\frac{1}{4}\frac{-1}{4}\rangle_{\rm RR}\\ \lvert\frac{1}{2}\frac{1}{2}\frac{-1}{4}\frac{1}{4}\rangle\lvert\frac{-1}{2}\frac{1}{2}\frac{1}{4}\frac{-1}{4}\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\emptyset$		$\emptyset$	$\emptyset$	$\emptyset$	
f.p.	$i,j\in\{1,2\}$ on $T_2\times T_3$	$(ij)\in\{(11),(12),(21),(22),$ $(13+14),(31+41),(23+24),(32+42),$ $(33+44),(34+43)\}$ on $T_2\times T_3$		$(ij)\in\{(i1\pm i1),(i2\pm i2),(i3\pm i4)\}$ on $T_1\times T_2$ with $i=1\ldots4$	(ijk) on $T_1\times T_2\times T_3$ with $i=1\ldots4;j,k=1,2$	$(ij)\in\{(i1\pm i1),(i2\pm i2),(i3\pm i4)\}$ on $T_1\times T_3$ with $i=1\ldots4$	
$(\#V,\#H)$	$(\frac{1+\eta}{2}\cdot8,\frac{1-\eta}{2}\cdot8)$	$(10,0)$		$(4+\frac{1+\eta}{2}\cdot8,0)$	$(\frac{1+\eta}{2}\cdot16,0)$	$(4+\frac{1+\eta}{2}\cdot8,0)$	
$\mathbb{Z}_2\times\mathbb{Z}_6$	$(0,\frac{1}{6},-\frac{1}{6})$	$(0,\frac{1}{3},-\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{3},-\frac{1}{6})$	$(\frac{1}{2},-\frac{1}{6},-\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$
V	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 00\frac{1}{6}\frac{5}{6}\rangle\lvert 00\frac{-1}{6}\frac{-5}{6}\rangle_{\rm NSNS}\\ \lvert 00\frac{-5}{6}\frac{-1}{6}\rangle\lvert 00\frac{5}{6}\frac{1}{6}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},\frac{1}{2},\frac{-1}{3},\frac{1}{3}\rangle\lvert\frac{1}{2},\frac{-1}{2},\frac{1}{3},\frac{-1}{3}\rangle_{\rm RR}\\ \lvert\frac{-1}{2},\frac{-1}{2},\frac{-1}{3},\frac{1}{3}\rangle\lvert\frac{-1}{2},\frac{1}{2},\frac{1}{3},\frac{-1}{3}\rangle_{\rm RR}\end{matrix}\right\}^{(11)}$	$\left\{\begin{matrix}\lvert 00\frac{1}{3}\frac{2}{3}\rangle\lvert 00\frac{-1}{3}\frac{-2}{3}\rangle_{\rm NSNS}\\ \lvert 00\frac{-2}{3}\frac{-1}{3}\rangle\lvert 00\frac{2}{3}\frac{1}{3}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},\frac{1}{2},\frac{-1}{6},\frac{1}{6}\rangle\lvert\frac{1}{2},\frac{-1}{2},\frac{1}{6},\frac{-1}{6}\rangle_{\rm RR}\\ \lvert\frac{-1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}\rangle\lvert\frac{-1}{2},\frac{1}{2},\frac{1}{6},\frac{-1}{6}\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle 00--\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert +-00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert -+00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle 0--0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert +00-\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert -00+\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\left\{\begin{matrix}\lvert 0,\frac{-1}{2},\frac{-1}{3},\frac{-1}{6}\rangle\lvert 0,\frac{1}{2},\frac{1}{3},\frac{1}{6}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},0,\frac{1}{6},\frac{1}{3}\rangle\lvert\frac{1}{2},0,\frac{-1}{6},\frac{-1}{3}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\left\{\begin{matrix}\lvert 0,\frac{-1}{2},\frac{-1}{6},\frac{-1}{3}\rangle\lvert 0,\frac{1}{2},\frac{1}{6},\frac{1}{3}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},0,\frac{1}{3},\frac{1}{6}\rangle\lvert\frac{1}{2},0,\frac{-1}{3},\frac{-1}{6}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle 0-0-\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert +0-0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert -0+0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$
H	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 00\frac{1}{6}\frac{5}{6}\rangle\lvert 00\frac{5}{6}\frac{1}{6}\rangle_{\rm NSNS}\\ \lvert 00\frac{-5}{6}\frac{-1}{6}\rangle\lvert 00\frac{-1}{6}\frac{-5}{6}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},\frac{1}{2},\frac{-1}{3},\frac{1}{3}\rangle\lvert\frac{-1}{2},\frac{1}{2},\frac{1}{3},\frac{-1}{3}\rangle_{\rm RR}\\ \lvert\frac{-1}{2},\frac{-1}{2},\frac{-1}{3},\frac{1}{3}\rangle\lvert\frac{1}{2}.\frac{-1}{2},\frac{1}{3},\frac{-1}{3}\rangle_{\rm RR}\end{matrix}\right\}^{(11)}$	$\left\{\begin{matrix}\lvert 00\frac{1}{3}\frac{2}{3}\rangle\lvert 00\frac{2}{3}\frac{1}{3}\rangle_{\rm NSNS}\\ \lvert 00\frac{-2}{3}\frac{-1}{3}\rangle\lvert 00\frac{-1}{3}\frac{-2}{3}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},\frac{1}{2},\frac{-1}{6},\frac{1}{6}\rangle\lvert\frac{-1}{2},\frac{1}{2},\frac{1}{6},\frac{-1}{6}\rangle_{\rm RR}\\ \lvert\frac{-1}{2},\frac{-1}{2},\frac{-1}{6},\frac{1}{6}\rangle\lvert\frac{1}{2},\frac{-1}{2},\frac{1}{6},\frac{-1}{6}\rangle_{\rm RR}\end{matrix}\right\}^{(kl)}$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00--\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert -+00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert +-00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0--0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert -00+\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert +00-\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\emptyset$	$\emptyset$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0-0-\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert -0+0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert +0-0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$
f.p.	(11) on $T_2\times T_3$	$(ij)\in\{(11),(12+13),(21+31),$ $(22+33+23+32)\}$ on $T_2\times T_3$ $(kl)=(22+33+23+32)$	$(ij)\in\{(11),(14+15+16),$ $(41+51+61),(44+56+65),$ $(45+54+66),(46+55+64)\}$ on $T_2\times T_3$	$(ij)\in\{(i1\pm i1),(i4+i5+i6)\}$ on $T_1\times T_2$ with $i=1,2,3,4$	$(ijk)\in\{(i11\pm i11),(i21\pm i31)\}$ on $T_1\times T_2\times T_3$ with $i=1,2,3,4$	$(ijk)\in\{(i11\pm i11),(i12\pm i13)\}$ on $T_1\times T_2\times T_3$ with $i=1,2,3,4$	$(ij)\in\{(i1\pm i1),(i4+i5+i6)\}$ on $T_1\times T_3$
$(\#V,\#H)$	$(\frac{1+\eta}{2}\cdot2,\frac{1-\eta}{2}\cdot2)$	$(8,2)$	$(\frac{1+\eta}{2}\cdot6,\frac{1-\eta}{2}\cdot6)$	$(\frac{1+\eta}{2}\cdot8,\frac{1-\eta}{2}\cdot4)$	$(4+\frac{1+\eta}{2}\cdot4,0)$	$(4+\frac{1+\eta}{2}\cdot4,0)$	$(\frac{1+\eta}{2}\cdot8,\frac{1-\eta}{2}\cdot4)$
$\mathbb{Z}_2\times\mathbb{Z}_6'$	$(-\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{6},-\frac{1}{3},\frac{1}{6})$	$(-\frac{1}{6},-\frac{1}{6},\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$
V	$\left\{\begin{matrix}\lvert 0,\frac{2}{3},\frac{1}{6},\frac{1}{6}\rangle\lvert 0,\frac{-2}{3},\frac{-1}{6},\frac{-1}{6}\rangle_{\rm NSNS}\\ \lvert\frac{-1}{2},\frac{1}{6},\frac{-1}{3},\frac{-1}{3}\rangle\lvert\frac{-1}{2},\frac{-1}{6},\frac{1}{3},\frac{1}{3}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\left\{\begin{matrix}\lvert 0,\frac{1}{3},\frac{1}{3},\frac{1}{3}\rangle\lvert 0,\frac{-1}{3},\frac{-1}{3},\frac{-1}{3}\rangle_{\rm NSNS}\\ \lvert\frac{-1}{2},\frac{-1}{6},\frac{-1}{6},\frac{-1}{6}\rangle\lvert\frac{-1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle 00--\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert +-00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert -+00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle 0--0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert +00-\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert -00+\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\left\{\begin{matrix}\lvert 0,\frac{1}{6},\frac{2}{3},\frac{1}{6}\rangle\lvert 0,\frac{-1}{6},\frac{-2}{3},\frac{-1}{6}\rangle_{\rm NSNS}\\ \lvert\frac{-1}{2},\frac{-1}{3},\frac{1}{6},\frac{-1}{3}\rangle\lvert\frac{-1}{2},\frac{1}{3},\frac{-1}{6},\frac{1}{3}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\left\{\begin{matrix}\lvert 0,-\frac{1}{6},-\frac{1}{6},-\frac{2}{3}\rangle\lvert 0,\frac{1}{6},\frac{1}{6},\frac{2}{3}\rangle_{\rm NSNS}\\ \lvert\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{-1}{6}\rangle\lvert\frac{1}{2},\frac{-1}{3},\frac{-1}{3},\frac{1}{6}\rangle_{\rm RR}\end{matrix}\right\}^{(ijk)}$	$\frac{1+\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle 0-0-\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert +0-0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert -0+0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$
H	$\emptyset$	$\emptyset$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 00++\rangle\lvert 00++\rangle_{\rm NSNS}\\ \lvert 00--\rangle\lvert 00--\rangle_{\rm NSNS}\\ \lvert ++00\rangle\lvert -+00\rangle_{\rm RR}\\ \lvert --00\rangle\lvert +-00\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0++0\rangle\lvert 0++0\rangle_{\rm NSNS}\\ \lvert 0--0\rangle\lvert 0--0\rangle_{\rm NSNS}\\ \lvert +00+\rangle\lvert -00+\rangle_{\rm RR}\\ \lvert -00-\rangle\lvert +00-\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$	$\emptyset$	$\emptyset$	$\frac{1-\eta}{2}\left\{\begin{matrix}\lvert 0+0+\rangle\lvert 0+0+\rangle_{\rm NSNS}\\ \lvert 0-0-\rangle\lvert 0-0-\rangle_{\rm NSNS}\\ \lvert +0+0\rangle\lvert -0+0\rangle_{\rm RR}\\ \lvert -0-0\rangle\lvert +0-0\rangle_{\rm RR}\end{matrix}\right\}^{(ij)}$
f.p.	$(ijk)\in\{(111\pm111),(211\pm311)\}$ on $T_1\times T_2\times T_3$	$(ijk)\in\{(111),(211+311),(121+131),(112+113),$ $(221+231+321+331),(212+213+312+313),$ $(122+133+132+123),(222+233+332+323),$ $(232+223+322+333)\}$ on $T_1\times T_2\times T_3$	$(ij)\in\{(11\pm11),(14+15+16),$ $(41+51+61),(44+55+66),(45+56+64),$ $(46+54+65)\}$ on $T_2\times T_3$	$(ij)\in\{(11\pm11),(14+15+16),$ $(41+51+61),(44+55+66),(45+56+64),$ $(46+54+65)\}$ on $T_1\times T_2$	$(ijk)\in\{(111\pm111),(121\pm131)\}$ on $T_1\times T_2\times T_3$	$(ijk)\in\{(111\pm111),(112\pm113)\}$ on $T_1\times T_2\times T_3$	$(ij)\in\{(11\pm11),(14+15+16),$ $(41+51+61),(44+55+66),(45+56+64),$ $(46+54+65)\}$ on $T_1\times T_3$
$(\#V,\#H)$	$(1+\frac{1+\eta}{2},0)$	$(9,0)$	$(\frac{1+\eta}{2}\cdot6,\frac{1-\eta}{2}\cdot5)$	$(\frac{1+\eta}{2}\cdot6,\frac{1-\eta}{2}\cdot5)$	$(1+\frac{1+\eta}{2},0)$	$(1+\frac{1+\eta}{2},0)$	$(\frac{1+\eta}{2}\cdot6,\frac{1-\eta}{2}\cdot5)$

Table 45: Massless closed string spectrum from the twisted sectors in $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ orbifolds without $(\eta=1)$ and with $(\eta=-1)$ discrete torsion. Only the NS-NS and RR sectors, which provide for spacetime bosons, are listed explicitly. The R-NS and NS-R sectors, which are spacetime fermions, can be infered by $\mathcal{N}=2$ supersymmetry in four dimensions or constructed explicitly in an analogous manner. All twisted sectors $n\vec{v}+m\vec{w}$ other than $\mathbb{Z}_2$ have inverse sectors $(N-n)\vec{v}+(M-m)\vec{w}$ which are not listed explicitly. The counting $(\#V,\#H)$ of $\mathcal{N}=2$ vector and hyper multiplets per sector includes these inverse sectors. The decomposition under the orientifold projection of the $\mathcal{N}=2$ vector multiplets into $\mathcal{N}=1$ vector and chiral multiplets is given in table 46.

Hodge numbers (h_{11}^+, h_{11}^-) per twist sector on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ without and with discrete torsion										
$T^6/$ torsion		Untwisted	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	$\vec{v}$	$(\vec{v}+\vec{w})$	$(\vec{v}+2\vec{w})$	$(\vec{v}+3\vec{w})$	total
$\mathbb{Z}_2 \times \mathbb{Z}_2$			$(0,\frac{1}{2},-\frac{1}{2})$			$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},0,-\frac{1}{2})$			
$\eta=1$	h_{11}^+ h_{11}^-	0 3	$8(b_2+b_3-b_2b_3)$ $8(2-b_2-b_3+b_2b_3)$			$8(b_1+b_2-b_1b_2)$ $8(2-b_1-b_2+b_1b_2)$	$8(b_1+b_3-b_1b_3)$ $8(2-b_1-b_3+b_1b_3)$			$16\sum_{i=1}^3 b_i - 8\sum_{i<j} b_ib_j$ $51-16\sum_{i=1}^3 b_i + 8\sum_{i<j} b_ib_j$
$\eta=-1$	h_{11}^+ h_{11}^-	0 3	0			0	0			0 3
$\mathbb{Z}_2 \times \mathbb{Z}_4$			$(0,\frac{1}{4},-\frac{1}{4})$	$(0,\frac{1}{2},-\frac{1}{2})$		$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{4},-\frac{1}{4})$	$(\frac{1}{2},0,-\frac{1}{2})$		
$\eta=1$	h_{11}^+ h_{11}^-	0 3	0 8	0 10		$6\,b$ $6(2-b)$	$8\,b$ $8(2-b)$	$6\,b$ $6(2-b)$		$20\,b$ $61-20\,b$
$\eta=-1$	h_{11}^+ h_{11}^-	0 3	0	0 10		**a/bAA, a/bAB**: $2\left[1-(1-b)\eta_{(3)}\right]$, $2\left[1+(1-b)\eta_{(3)}\right]$ **a/bBB**: $2\left[1+(1-b)\eta_{(3)}\right]$, $2\left[1-(1-b)\eta_{(3)}\right]$	0	**a/bAA**: $2\left[1-(1-b)\eta_{(3)}\right]$, $2\left[1+(1-b)\eta_{(3)}\right]$ **a/bAB, a/bBB**: $2\left[1+(1-b)\eta_{(3)}\right]$, $2\left[1-(1-b)\eta_{(3)}\right]$		**a/bAA**: $4\left[1-(1-b)\eta_{(3)}\right]$, $17+4(1-b)\eta_{(3)}$ **a/bAB**: 4, 17 **a/bBB**: $4\left[1+(1-b)\eta_{(3)}\right]$, $17-4(1-b)\eta_{(3)}$
$\mathbb{Z}_2 \times \mathbb{Z}_6$			$(0,\frac{1}{6},-\frac{1}{6})$	$(0,\frac{1}{3},-\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{3},-\frac{1}{6})$	$(\frac{1}{2},-\frac{1}{6},-\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
$\eta=1$	h_{11}^+ h_{11}^-	0 3	0 2	0 8	1 5	$4\,b$ $4(2-b)$	$4\,b$ $4(2-b)$	$4\,b$ $4(2-b)$	$4\,b$ $4(2-b)$	$1+16\,b$ $50-16\,b$
$\eta=-1$	h_{11}^+ h_{11}^-	0 3	0	0 8	0	0	**a/bAA, a/bAB**: $2\left[1+(1-b)\eta_{(2)}\right]$, $2\left[1-(1-b)\eta_{(2)}\right]$ **a/bBB**: $2\left[1-(1-b)\eta_{(2)}\right]$, $2\left[1+(1-b)\eta_{(2)}\right]$	**a/bAA**: $2\left[1+(1-b)\eta_{(3)}\right]$, $2\left[1-(1-b)\eta_{(3)}\right]$ **a/bAB, a/bBB**: $2\left[1-(1-b)\eta_{(3)}\right]$, $2\left[1+(1-b)\eta_{(3)}\right]$	0	**a/bAA**: $4+2(1-b)(\eta_{(2)}+\eta_{(3)})$, $15-2(1-b)(\eta_{(2)}+\eta_{(3)})$ **a/bAB**: $4+2(1-b)(\eta_{(2)}-\eta_{(3)})$, $15-2(1-b)(\eta_{(2)}-\eta_{(3)})$ **a/bBB**: $4-2(1-b)(\eta_{(2)}+\eta_{(3)})$, $15+2(1-b)(\eta_{(2)}+\eta_{(3)})$
$\mathbb{Z}_2 \times \mathbb{Z}_6'$			$(-\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2},0)$	$(\frac{1}{6},-\frac{1}{3},\frac{1}{6})$	$(-\frac{1}{6},-\frac{1}{6},\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
$\eta=1$	h_{11}^+ h_{11}^-	0 3	0 2	**AAA, ABB**: 1, 8 **AAB, BBB**: 0, 9	1 5	1 5	0 2	0 2	1 5	**AAA, ABB**: 4, 32 **AAB, BBB**: 3, 33
$\eta=-1$	h_{11}^+ h_{11}^-	0 3	**AAA, AAB, ABB**: $\frac{1+\eta_{(1)}}{2}$, $\frac{1-\eta_{(1)}}{2}$ **BBB**: $\frac{1-\eta_{(1)}}{2}$, $\frac{1+\eta_{(1)}}{2}$	**AAA, ABB**: 1, 8 **AAB, BBB**: 0, 9	0	0	**AAA, AAB**: $\frac{1+\eta_{(2)}}{2}$, $\frac{1-\eta_{(2)}}{2}$ **ABB, BBB**: $\frac{1-\eta_{(2)}}{2}$, $\frac{1+\eta_{(2)}}{2}$	**AAA**: $\frac{1+\eta_{(3)}}{2}$, $\frac{1-\eta_{(3)}}{2}$ **AAB, ABB, BBB**: $\frac{1-\eta_{(3)}}{2}$, $\frac{1+\eta_{(3)}}{2}$	0	**AAA**: $\frac{5+\sum_{i=1}^3 \eta_{(i)}}{2}$, $\frac{25-\sum_{i=1}^3 \eta_{(i)}}{2}$ **AAB**: $\frac{3+\eta_{(1)}+\eta_{(2)}-\eta_{(3)}}{2}$, $\frac{27-\eta_{(1)}-\eta_{(2)}+\eta_{(3)}}{2}$ **ABB**: $\frac{5+\eta_{(1)}-\eta_{(2)}-\eta_{(3)}}{2}$, $\frac{25-\eta_{(1)}+\eta_{(2)}+\eta_{(3)}}{2}$ **BBB**: $\frac{3-\sum_{i=1}^3 \eta_{(i)}}{2}$, $\frac{27+\sum_{i=1}^3 \eta_{(i)}}{2}$

Table 46: Decomposition of Hodge number $h_{11} = h_{11}^+ + h_{11}^-$ under the orientifold projection per twist-sector for orbifolds $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_{2M} \times \Omega\mathcal{R})$ without and with torsion. h_{11}^- counts the Kähler moduli, whereas h_{11}^+ labels the number of Abelian vectors.

Massless open string states				
$\frac{\text{angle}}{\pi}$	NS-sector	R-sector	$(c^{\mathbb{Z}_2^{(1)}}, c^{\mathbb{Z}_2^{(2)}}, c^{\mathbb{Z}_2^{(3)}})$	representation
$(0,0,0)$	$\psi^{\mu}_{-1/2}\|0\rangle_{\mathrm{NS}}$	$\|0\rangle_{\mathrm{R}}, \psi_0^{\mu}\prod_{i=1}^{3}\psi_0^i\|0\rangle_{\mathrm{R}}$	$(+,+,+)$	$\prod_{k=1}^{4} U(N^k)$
	$\psi^{1}_{-1/2}\|0\rangle_{\mathrm{NS}}, \psi^{\overline{1}}_{-1/2}\|0\rangle_{\mathrm{NS}}$	$\psi_0^2\psi_0^3\|0\rangle_{\mathrm{R}}, \psi_0^{\mu}\psi_0^1\|0\rangle_{\mathrm{R}}$	$(+,-,-)$	$(\mathbf{N}^1,\overline{\mathbf{N}}^2)+(\mathbf{N}^3,\overline{\mathbf{N}}^4)+c.c.$
	$\psi^{2}_{-1/2}\|0\rangle_{\mathrm{NS}}, \psi^{\overline{2}}_{-1/2}\|0\rangle_{\mathrm{NS}}$	$\psi_0^1\psi_0^3\|0\rangle_{\mathrm{R}}, \psi_0^{\mu}\psi_0^2\|0\rangle_{\mathrm{R}}$	$(-,+,-)$	$(\mathbf{N}^1,\overline{\mathbf{N}}^3)+(\mathbf{N}^2,\overline{\mathbf{N}}^4)+c.c.$
	$\psi^{3}_{-1/2}\|0\rangle_{\mathrm{NS}}, \psi^{\overline{3}}_{-1/2}\|0\rangle_{\mathrm{NS}}$	$\psi_0^1\psi_0^2\|0\rangle_{\mathrm{R}}, \psi_0^{\mu}\psi_0^3\|0\rangle_{\mathrm{R}}$	$(-,-,+)$	$(\mathbf{N}^1,\overline{\mathbf{N}}^4)+(\mathbf{N}^2,\overline{\mathbf{N}}^3)+c.c.$
$(0,\phi,-\phi)$	$\psi^{3}_{-1/2+\phi}\|0\rangle_{\mathrm{NS}}^{(\mathrm{tw})}$	$\|\tilde{0}\rangle_{\mathrm{R}}^{(\mathrm{tw},1)}$	$(-,-,+)$	$\left\{\begin{array}{l}(\mathbf{N}^1,\overline{\mathbf{N}}^4)+(\mathbf{N}^2,\overline{\mathbf{N}}^3)\\ +(\mathbf{N}^3,\overline{\mathbf{N}}^2)+(\mathbf{N}^4,\overline{\mathbf{N}}^1)\end{array}\right.$
	$\psi^{\overline{2}}_{-1/2+\phi}\|0\rangle_{\mathrm{NS}}^{(\mathrm{tw})}$	$\psi_0^{\mu}\psi_0^1\|\tilde{0}\rangle_{\mathrm{R}}^{(\mathrm{tw},1)}$	$(-,+,-)$	$\left\{\begin{array}{l}(\overline{\mathbf{N}}^1,\mathbf{N}^3)+(\overline{\mathbf{N}}^2,\mathbf{N}^4)\\ +(\overline{\mathbf{N}}^3,\mathbf{N}^1)+(\overline{\mathbf{N}}^4,\mathbf{N}^2)\end{array}\right.$
$(\phi_1,\phi_2,\phi_3)_{\sum_i\phi_i=0}$ $\phi_2,\phi_3>0,\phi_1<0$ $\|\phi_1\|>\|\phi_2\|,\|\phi_3\|$	$\psi^{1}_{-1/2+\phi_1}\|0\rangle_{\mathrm{NS}}^{(\mathrm{tw})}$	$\|\tilde{0}\rangle_{\mathrm{R}}^{(\mathrm{tw})}$	$(+,-,-)$	$(\mathbf{N}^1,\overline{\mathbf{N}}^2)+(\mathbf{N}^3,\overline{\mathbf{N}}^4)$

Table 47: Counting of massless open string states, their $\mathbb{Z}_2^{(i)}$ eigenvalues on $T^6/\mathbb{Z}_2\times\mathbb{Z}_{2M}$ with discrete torsion and representations computed from Chan-Paton labels. The multiplets from D6-branes at angles have chiralities depending on the values of the angles which in turn determine the massless R-sector states. The representations in the last column correspond to full $\mathcal{N}=1$ multiplets, where for some non-vanishing angle, the explicitly listed NS and R states have been paired up with those from the inverse angles, e.g. $\pm(0,\phi,-\phi)$.

$b_{SU(N_a)}$ **for bifundamental and adjoints:** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ **without and with discrete torsion**		
$\frac{\text{Angle}}{\pi}$	$b_{SU(N_a)}^{\text{no torsion}}$	$b_{SU(N_a)}^{\text{with torsion}}$
$(0,0,0)$	$-$	$-N_b \left(\prod_{n=1}^3 \delta_{\sigma_n^{ab},0}\delta_{\tau_n^{ab},0}\right) \sum_{i=1}^3 (-1)^{\tau_{ab}^{\mathbb{Z}_2^{(i)}}}$
$(0,\phi,-\phi)$	$N_b\, \delta_{\sigma_1^{ab},0}\, \delta_{\tau_1^{ab},0}\, \lvert I_{ab}^{(2\cdot 3)}\rvert$	$\frac{N_b}{4}\, \delta_{\sigma_1^{ab},0}\, \delta_{\tau_1^{ab},0} \left(\lvert I_{ab}^{(2\cdot 3)}\rvert - I_{ab}^{\mathbb{Z}_2^{(1)},(2\cdot 3)}\right)$
$(\phi^{(1)},\phi^{(2)},\phi^{(3)})_{\sum_{n=1}^3 \phi^{(n)}=0}$	$\frac{N_b}{2}\, \lvert I_{ab}\rvert$	$\frac{N_b}{8} \left(\lvert I_{ab}\rvert + \text{sgn}(I_{ab}) \sum_{i=1}^3 I_{ab}^{\mathbb{Z}_2^{(i)}}\right)$

Table 48: Counting bifundamental and adjoint representations using their beta function coefficients. By comparison with (138), in the case *without discrete torsion* one obtains three multiplets in the adjoint representation from the aa sector and for $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$ and $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ without torsion additionally $k \times |I_{a(\theta^k a)}^{(2\cdot 3)}|$ adjoint multiplets, where $k=1$ for the former and $k=2$ for the latter. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ without discrete torsion, $|I_{a(\theta^k a)}|$ multiplets in the adjoint representation parameterise the freedom for brane recombination of orbifold images. In the presence of *discrete torsion*, as anticipated in table 47, the aa sector only contains the vector of $U(N_a)$, and in the absence of relative discrete Wilson lines and displacements, parallel branes ab with different $\mathbb{Z}_2^{(i)}$ eigenvalues support one non-chiral pair of multiplets in the bifundamental representation. The symbols σ_i^{ab}, τ_i^{ab}, $\tau_{ab}^{\mathbb{Z}_2^i}$ are related to relative quantities between brane a and b: relative displacement, relative Wilson line along the i^{th} T^2 and the relative eigenvalue under $\mathbb{Z}_2^{(i)}$. ('Related to' means that they, as always, appear as powers of (-1).)

$b_{SU(N_a)}$ **for (anti)symmetrics:** $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ **without and with discrete torsion**		
$\frac{\text{Angle}}{\pi}$	$b_{SU(N_a)}^{\text{no torsion}}$	$b_{SU(N_a)}^{\text{with torsion}}$
$(0,0,0)$ $\uparrow\uparrow \Omega\mathcal{R}$	$\sum_{i=1}^{3} \delta_{\sigma_i^{aa'},0}\delta_{\tau_i^{aa'},0}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)},(j\cdot k)}$	$-\frac{N_a}{4}\, \sum_{i=1}^{3} I_{aa'}^{\mathbb{Z}_2^{(i)},(j\cdot k)}$ $+\frac{1}{2}\sum_{i=1}^{3} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)},(j\cdot k)}$
$(0,0,0)$ $\uparrow\uparrow \Omega\mathcal{R}\mathbb{Z}_2^{(i)}$	$\delta_{\sigma_i^{aa'},0}\delta_{\tau_i^{aa'},0}\, \tilde{I}_a^{\Omega\mathcal{R},(j\cdot k)}$ $+\sum_{j\neq i} \delta_{\sigma_k^{aa'},0}\delta_{\tau_k^{aa'},0}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(j)},(i\cdot j)}$	$-N_a\, \sum_{i=1}^{3}(-1)^{\tau_{aa'}^{\mathbb{Z}_2^{(i)}}}$ $+\frac{1}{2}\left(\eta_{\Omega\mathcal{R}}\, \tilde{I}_a^{\Omega\mathcal{R},(j\cdot k)} + \sum_{j\neq i} \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(j)}}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(j)},(i\cdot j)}\right)$
$(0_i,\phi_j,\phi_k)_{\phi_k=-\phi_j\neq\pm\frac{1}{2}}$ $\uparrow\uparrow\left(\Omega\mathcal{R}+\Omega\mathcal{R}\mathbb{Z}_2^{(i)}\right)$	$\delta_{\sigma_i^{aa'}}\delta_{\tau_i^{aa'},0}\left\{N_a\,\lvert I_{aa'}^{(j\cdot k)}\rvert \right.$ $\left. +\tilde{I}_a^{\Omega\mathcal{R},(j\cdot k)} + \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)},(j\cdot k)}\right\}$	$\frac{N_a}{4}\, \left(\lvert I_{aa'}^{(j\cdot k)}\rvert - I_{aa'}^{\mathbb{Z}_2^{(i)},(j\cdot k)}\right)$ $+\frac{1}{2}\left(\eta_{\Omega\mathcal{R}}\, \tilde{I}_a^{\Omega\mathcal{R},(j\cdot k)} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)},(j\cdot k)}\right)$
$(0_i,\phi_j,\phi_k)_{\phi_k=-\phi_j\neq\pm\frac{1}{2}}$ $\uparrow\uparrow\left(\Omega\mathcal{R}\mathbb{Z}_2^{(j)}+\Omega\mathcal{R}\mathbb{Z}_2^{(k)}\right)$	$\delta_{\sigma_i^{aa'},0}\delta_{\tau_i^{aa'},0}\left\{N_a\,\lvert I_{aa'}^{(j\cdot k)}\rvert \right.$ $\left. +\tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(j)},(j\cdot k)} + \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(k)},(j\cdot k)}\right\}$	$\frac{N_a}{4}\, \left(\lvert I_{aa'}^{(j\cdot k)}\rvert - I_{aa'}^{\mathbb{Z}_2^{(i)},(j\cdot k)}\right)$ $+\frac{1}{2}\left(\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(j)}}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(j)},(j\cdot k)} + \eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(k)}}\, \tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(k)},(j\cdot k)}\right)$
$(\phi^{(1)},\phi^{(2)},\phi^{(3)})$ $\sum_{n=1}^{3}\phi^{(n)}=0$	$\frac{N_a}{2}\,\lvert I_{aa'}\rvert$ $+\frac{c_a^{\Omega\mathcal{R}}1}{2}\tilde{I}_a^{\Omega\mathcal{R}} + \sum_{i=1}^{3}\frac{c_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}}{2}\tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}$	$\frac{N_a}{8}\, \left(\lvert I_{aa'}\rvert + \mathrm{sgn}(I_{aa'})\sum_{i=1}^{3} I_{aa'}^{\mathbb{Z}_2^{(i)}}\right)$ $+\frac{1}{4}\left(c_a^{\Omega\mathcal{R}}\,\eta_{\Omega\mathcal{R}}\,\tilde{I}_a^{\Omega\mathcal{R}} + \sum_{i=1}^{3} c_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\,\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\,\tilde{I}_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}\right)$

Table 49: Counting symmetric and antisymmetric matter states. The constants $c_a^{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} = \mathrm{sgn}(\tilde{\phi}^{(k)})\left[2H(|\tilde{\phi}^{(k)}|-\frac{1}{2})-1\right] \in \{-1,0,1\}$ arise from Möbius strip contributions when the D6$_a$-branes are at three non-vanishing angles with the $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ planes preserving the aa' open string sector. $0<|\tilde{\phi}^{(k)}|<1$ is the largest absolute value of the three angles between the D6$_a$-brane and the $\Omega\mathcal{R}\mathbb{Z}_2^{(i)}$ plane, and $H(x)=0,\frac{1}{2},1$ for $x<0, x=0, x>0$, respectively, is the Heavyside step function. Note that the Kronecker deltas $\delta_{\sigma_i^{aa'},0}\delta_{\tau_i^{aa'},0}$ can be dropped in the expressions with discrete torsion due to the discrete nature of the displacements and Wilson lines. For the Möbius strip amplitude on a tilted torus, however, we run into an inconsistency in the classification of orientifold invariant D6$_c$-branes, unless we replace e.g. $\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}} \to (-1)^{2b_i\sigma_i^c\tau_i^c}\eta_{\Omega\mathcal{R}\mathbb{Z}_2^{(i)}}$ in the formula for the beta function coefficient of a D6$_c$-brane parallel to the $\Omega\mathcal{R}$ invariant plane. We expect that a thorough re-examination of the lattice contributions, which goes beyond the scope of the present work, will provide the missing sign factor.

$\Omega\mathcal{R}$ images of $\mathbb{Z}_2^{(k)}$ fixed points on $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ and classification of $\Omega\mathcal{R}$ invariant three-cycles				
T_i T_j T_k	$\mathbb{Z}_2^{(k)}$ fixed points on $T_i\times T_j : (x_1y_1\vert x_2y_1\vert x_1y_2\vert x_2y_2)$			
(σ_i,σ_j)	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$\rightarrow$ $\rightarrow$ $\rightarrow$ $\Omega\mathcal{R}$	$(1,1\vert\frac{2}{1-b_i},1\vert 1,\frac{2}{1-b_j}\vert\frac{2}{1-b_i},\frac{2}{1-b_j})$ $(1,1\vert\frac{2}{1-b_i},1\vert 1,\frac{2}{1-b_j}\vert\frac{2}{1-b_i},\frac{2}{1-b_j})$	$(4(1-b_i),1\vert 3,1\vert 4(1-b_i),\frac{2}{1-b_j}\vert 3,\frac{2}{1-b_j})$ $(4-2b_i,1\vert 2+2b_i,1\vert 4-2b_i,\frac{2}{1-b_j}\vert 3-2b_i,\frac{2}{1-b_j})$	$(1,4(1-b_j)\vert\frac{2}{1-b_i},4(1-b_j)\vert 1,3\vert\frac{2}{1-b_i},3)$ $(1,4-2b_j\vert\frac{2}{1-b_i},3-2b_j\vert 1,4-2b_j\vert\frac{2}{1-b_i},3-2b_j)$	$(4(1-b_i),4(1-b_j)\vert 3,4(1-b_j)\vert 4(1-b_i),3\vert 3,3)$ $(4-2b_i,4-2b_j\vert 3-2b_i,4-2b_j\vert 4-2b_i,3-2b_j\vert 3-2b_i,3-2b_j)$
aa 3-cycle invariant for	all sets pointwise invariant $\eta_{(k)}=-1$			
ab 3-cycle invariant for	sets pointwise invariant $\eta_{(k)}=-1$		permututation within sets $(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=-(-1)^{\tau_j}$	
ba 3-cycle invariant for	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$
bb 3-cycle invariant for	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$	$(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=-(-1)^{\tau_j}$	$(x_2y_2\vert x_1y_2\vert x_2y_1\vert x_1y_1)$ $\eta_{(k)}=-(-1)^{\tau_i+\tau_j}$
3-cycle invariant for	$\eta_{(k)}=-(-1)^{2(b_i\sigma_i\tau_i+b_j\sigma_j\tau_j)}$			
$\uparrow$ $\downarrow$ $\rightarrow$ $\Omega\mathcal{R}$	$(1,1\vert 4,1\vert 1,4\vert 4,4)$ $(1,1\vert 4,1\vert 1,4\vert 4,4)$	$(2,1\vert 3,1\vert 2,4\vert 3,4)$ $(2+2b_i,1\vert 3-2b_i,1\vert 2+2b_i,4\vert 3-2b_i,4)$	$(1,2\vert 4,2\vert 1,3\vert 4,3)$ $(1,2+2b_j\vert 4,2+2b_j\vert 1,3-2b_j\vert 4,3-2b_j)$	$(2,2\vert 3,2\vert 2,3\vert 3,3)$ $(2+2b_i,2+2b_j\vert 3-2b_i,2+2b_j\vert 2+2b_i,3-2b_j\vert 3-2b_i,3-2b_j)$
aa 3-cycle invariant for	all sets pointwise invariant $\eta_{(k)}=-1$			
ab 3-cycle invariant for	sets pointwise invariant $\eta_{(k)}=-1$		permututation within sets $(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=-(-1)^{\tau_j}$	
ba 3-cycle invariant for	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$
bb 3-cycle invariant for	pointwise invariant $\eta_{(k)}=-1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=-(-1)^{\tau_i}$	$(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=-(-1)^{\tau_j}$	$(x_2y_2\vert x_1y_2\vert x_2y_1\vert x_1y_1)$ $\eta_{(k)}=-(-1)^{\tau_i+\tau_j}$
3-cycle invariant for	$\eta_{(k)}=-(-1)^{2(b_i\sigma_i\tau_i+b_j\sigma_j\tau_j)}$			
$\uparrow$ $\rightarrow$ $\downarrow$ $\Omega\mathcal{R}$	$(1,1\vert 4,1\vert 1,\frac{2}{1-b_j}\vert 4,\frac{2}{1-b_j})$ $(1,1\vert 4,1\vert 1,\frac{2}{1-b_j}\vert 4,\frac{2}{1-b_j})$	$(2,1\vert 3,1\vert 2,\frac{2}{1-b_j}\vert 3,\frac{2}{1-b_j})$ $(2+2b_i,1\vert 3-2b_i,1\vert 2+2b_i,\frac{2}{1-b_j}\vert 3-2b_i,\frac{2}{1-b_j})$	$(1,4(1-b_j)\vert 4,4(1-b_j)\vert 1,3\vert 4,3)$ $(1,4-2b_j\vert 4,4-2b_j\vert 1,3-2b_j\vert 4,3-2b_j)$	$(2,4(1-b_j)\vert 3,4(1-b_j)\vert 2,3\vert 3,3)$ $(2+2b_i,4-2b_j\vert 3-2b_i,4-2b_j\vert 2+2b_i,3-2b_j\vert 3-2b_i,3-2b_j)$
aa 3-cycle invariant for	all sets pointwise invariant $\eta_{(k)}=1$			
ab 3-cycle invariant for	sets pointwise invariant $\eta_{(k)}=1$		permututation within sets $(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=(-1)^{\tau_j}$	
ba 3-cycle invariant for	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$
bb 3-cycle invariant for	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$	$(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=(-1)^{\tau_j}$	$(x_2y_2\vert x_1y_2\vert x_2y_1\vert x_1y_1)$ $\eta_{(k)}=(-1)^{\tau_i+\tau_j}$
3-cycle invariant for	$\eta_{(k)}=(-1)^{2(b_i\sigma_i\tau_i+b_j\sigma_j\tau_j)}$			
$\rightarrow$ $\uparrow$ $\downarrow$ $\Omega\mathcal{R}$	$(1,1\vert\frac{2}{1-b_i},1\vert 1,4\vert\frac{2}{1-b_i},4)$ $(1,1\vert\frac{2}{1-b_i},1\vert 1,4\vert\frac{2}{1-b_i},4)$	$(4(1-b_i),1\vert 3,1\vert 4(1-b_i),4\vert 3,4)$ $(4-2b_i,1\vert 3-2b_i,1\vert 4-2b_i,4\vert 3-2b_i,4)$	$(1,2\vert\frac{2}{1-b_i},2\vert 1,3\vert\frac{2}{1-b_i},3)$ $(1,2+2b_i\vert\frac{2}{1-b_i},2+2b_i\vert 1,3-2b_i\vert\frac{2}{1-b_i},3-2b_i)$	$(4(1-b_i),2\vert 3,2\vert 4(1-b_i),3\vert 3,3)$ $(4-2b_i,2+2b_j\vert 3-2b_i,2+2b_j\vert 4-2b_i,3-2b_j\vert 3-2b_i,3-2b_j)$
aa 3-cycle invariant for	all sets pointwise invariant $\eta_{(k)}=1$			
ab 3-cycle invariant for	sets pointwise invariant $\eta_{(k)}=1$		permututation within sets $(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=(-1)^{\tau_j}$	
ba 3-cycle invariant for	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$
bb 3-cycle invariant for	pointwise invariant $\eta_{(k)}=1$	$(x_2y_1\vert x_1y_1\vert x_2y_2\vert x_1y_2)$ $\eta_{(k)}=(-1)^{\tau_i}$	$(x_1y_2\vert x_2y_2\vert x_1y_1\vert x_2y_1)$ $\eta_{(k)}=(-1)^{\tau_j}$	$(x_2y_2\vert x_1y_2\vert x_2y_1\vert x_1y_1)$ $\eta_{(k)}=(-1)^{\tau_i+\tau_j}$
3-cycle invariant for	$\eta_{(k)}=(-1)^{2(b_i\sigma_i\tau_i+b_j\sigma_j\tau_j)}$			

Table 50: Transformation of $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ fixed points in the $\mathbb{Z}_2^{(k)}$ twisted sector and conditions for $\Omega\mathcal{R}$ invariance of the three-cycle in dependence of the choice of the exotic O6-plane.

Existence of $Sp(2M)$ or $SO(2M)$ gauge factors on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orientifolds with discrete torsion				
3-cycle ↑↑ to O6-plane	$(\eta_{(1)}, \eta_{(2)}, \eta_{(3)})$			
$(\sigma_1, \sigma_2, \sigma_3)$	$(-1,-1,-1)$	$(-1,1,1)$	$(1,-1,1)$	$(1,1,-1)$
$\Omega\mathcal{R}$	$b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_1\sigma_1\tau_1 \neq b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_2\sigma_2\tau_2 \neq b_1\sigma_1\tau_1 = b_3\sigma_3\tau_3$	$b_3\sigma_3\tau_3 \neq b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2$
$(0,0,0)$	any choice of $b_i\tau_i$	$\times$	$\times$	$\times$
$(1,0,0)$	$b_1\tau_1 = 0$	$2b_1\tau_1 = 1$	$\times$	$\times$
$(0,1,0)$	$b_2\tau_2 = 0$	$\times$	$2b_2\tau_2 = 1$	$\times$
$(0,0,1)$	$b_3\tau_3 = 0$	$\times$	$\times$	$2b_3\tau_3 = 1$
$(1,1,0)$	$b_1\tau_1 = b_2\tau_2 = 0$	$2b_1\tau_1 = 1; b_2\tau_2 = 0$	$b_1\tau_1 = 0; 2b_2\tau_2 = 1$	$2b_1\tau_1 = 2b_2\tau_2 = 1$
$(1,0,1)$	$b_1\tau_1 = b_3\tau_3 = 0$	$2b_1\tau_1 = 1; b_3\tau_3 = 0$	$2b_1\tau_1 = 2b_3\tau_3 = 1$	$b_1\tau_1 = 0; 2b_3\tau_3 = 1$
$(0,1,1)$	$b_2\tau_2 = b_3\tau_3 = 0$	$2b_2\tau_2 = 2b_3\tau_3 = 1$	$2b_2\tau_2 = 1; b_3\tau_3 = 0$	$b_2\tau_2 = 0; 2b_3\tau_3 = 1$
$(1,1,1)$	$b_1\tau_1 = b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 \neq b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 = b_3\tau_3 \neq b_2\tau_2$	$b_1\tau_1 = b_2\tau_2 \neq b_3\tau_3$
$\Omega\mathcal{R}\mathbb{Z}_2^{(1)}$	$b_1\sigma_1\tau_1 \neq b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_3\sigma_3\tau_3 \neq b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2$	$b_2\sigma_2\tau_2 \neq b_1\sigma_1\tau_1 = b_3\sigma_3\tau_3$
$(0,0,0)$	$\times$	any choice of $b_i\tau_i$	$\times$	$\times$
$(1,0,0)$	$2b_1\tau_1 = 1$	$b_1\tau_1 = 0$	$\times$	$\times$
$(0,1,0)$	$\times$	$b_2\tau_2 = 0$	$\times$	$2b_2\tau_2 = 1$
$(0,0,1)$	$\times$	$b_3\tau_3 = 0$	$2b_3\tau_3 = 1$	$\times$
$(1,1,0)$	$2b_1\tau_1 = 1; b_2\tau_2 = 0$	$b_1\tau_1 = b_2\tau_2 = 0$	$2b_1\tau_1 = 2b_2\tau_2 = 1$	$b_1\tau_1 = 0; 2b_2\tau_2 = 1$
$(1,0,1)$	$2b_1\tau_1 = 1; b_3\tau_3 = 0$	$b_1\tau_1 = b_3\tau_3 = 0$	$b_1\tau_1 = 0; 2b_3\tau_3 = 1$	$2b_1\tau_1 = 2b_3\tau_3 = 1$
$(0,1,1)$	$2b_2\tau_2 = 2b_3\tau_3 = 1$	$b_2\tau_2 = b_3\tau_3 = 0$	$b_2\tau_2 = 0; 2b_3\tau_3 = 1$	$2b_2\tau_2 = 1; b_3\tau_3 = 0$
$(1,1,1)$	$b_1\tau_1 \neq b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 = b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 = b_2\tau_2 \neq b_3\tau_3$	$b_1\tau_1 = b_3\tau_3 \neq b_2\tau_2$
$\Omega\mathcal{R}\mathbb{Z}_2^{(2)}$	$b_2\sigma_2\tau_2 \neq b_1\sigma_1\tau_1 = b_3\sigma_3\tau_3$	$b_3\sigma_3\tau_3 \neq b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2$	$b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_1\sigma_1\tau_1 \neq b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$
$(0,0,0)$	$\times$	$\times$	any choice of $b_i\tau_i$	$\times$
$(1,0,0)$	$\times$	$\times$	$b_1\tau_1 = 0$	$2b_1\tau_1 = 1$
$(0,1,0)$	$2b_2\tau_2 = 1$	$\times$	$b_2\tau_2 = 0$	$\times$
$(0,0,1)$	$\times$	$2b_3\tau_3 = 1$	$b_3\tau_3 = 0$	$\times$
$(1,1,0)$	$b_1\tau_1 = 0; 2b_2\tau_2 = 1$	$2b_1\tau_1 = 2b_2\tau_2 = 1$	$b_1\tau_1 = b_2\tau_2 = 0$	$2b_1\tau_1 = 1; b_2\tau_2 = 0$
$(1,0,1)$	$2b_1\tau_1 = 2b_3\tau_3 = 1$	$b_1\tau_1 = 0; 2b_3\tau_3 = 1$	$b_1\tau_1 = b_3\tau_3 = 0$	$2b_1\tau_1 = 1; b_3\tau_3 = 0$
$(0,1,1)$	$2b_2\tau_2 = 1; b_3\tau_3 = 0$	$b_2\tau_2 = 0; 2b_3\tau_3 = 1$	$b_2\tau_2 = b_3\tau_3 = 0$	$2b_2\tau_2 = 2b_3\tau_3 = 1$
$(1,1,1)$	$b_1\tau_1 = b_3\tau_3 \neq b_2\tau_2$	$b_1\tau_1 = b_2\tau_2 \neq b_3\tau_3$	$b_1\tau_1 = b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 \neq b_2\tau_2 = b_3\tau_3$
$\Omega\mathcal{R}\mathbb{Z}_2^{(3)}$	$b_3\sigma_3\tau_3 \neq b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2$	$b_2\sigma_2\tau_2 \neq b_1\sigma_1\tau_1 = b_3\sigma_3\tau_3$	$b_1\sigma_1\tau_1 \neq b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$	$b_1\sigma_1\tau_1 = b_2\sigma_2\tau_2 = b_3\sigma_3\tau_3$
$(0,0,0)$	$\times$	$\times$	$\times$	any choice of $b_i\tau_i$
$(1,0,0)$	$\times$	$\times$	$2b_1\tau_1 = 1$	$b_1\tau_1 = 0$
$(0,1,0)$	$\times$	$2b_2\tau_2 = 1$	$\times$	$b_2\tau_2 = 0$
$(0,0,1)$	$2b_3\tau_3 = 1$	$\times$	$\times$	$b_3\tau_3 = 0$
$(1,1,0)$	$2b_1\tau_1 = 2b_2\tau_2 = 1$	$b_1\tau_1 = 0; 2b_2\tau_2 = 1$	$2b_1\tau_1 = 1; b_2\tau_2 = 0$	$b_1\tau_1 = b_2\tau_2 = 0$
$(1,0,1)$	$b_1\tau_1 = 0; 2b_3\tau_3 = 1$	$2b_1\tau_1 = 2b_3\tau_3 = 1$	$2b_1\tau_1 = 1; b_3\tau_3 = 0$	$b_1\tau_1 = b_3\tau_3 = 0$
$(0,1,1)$	$b_2\tau_2 = 0; 2b_3\tau_3 = 1$	$2b_2\tau_2 = 1; b_3\tau_3 = 0$	$2b_2\tau_2 = 2b_3\tau_3 = 1$	$b_2\tau_2 = b_3\tau_3 = 0$
$(1,1,1)$	$b_1\tau_1 = b_2\tau_2 \neq b_3\tau_3$	$b_1\tau_1 = b_3\tau_3 \neq b_2\tau_2$	$b_1\tau_1 \neq b_2\tau_2 = b_3\tau_3$	$b_1\tau_1 = b_2\tau_2 = b_3\tau_3$

Table 51: Existence of $\Omega\mathcal{R}$ invariant three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_{2M}$ orientifolds with discrete torsion. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, all choices of b_i correspond to untilted or tilted tori. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$, the second and third two torus are tilted, $b_2 = b_3 = 1/2$, but the shape of the first two torus remains a free parameter, $b_1 \equiv b$. For $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$, all tori are tilted, i.e. $b_1 = b_2 = b_3 = 1/2$.

f.p.$^{(1)}$	orbit	f.p.$^{(1)}$	orbit	f.p.$^{(1)}$	orbit	f.p.$^{(1)}$	orbit
$(n^2,m^2;n^3,m^3) = $ (odd,odd;odd,odd)							
$(\sigma_2;\sigma_3) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$	
11 61 16 66	$[n^1\varepsilon_0^{(1)} + m^1\tilde{\varepsilon}_0^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$	41 51 46 56	$[1+(-1)^{\tau_2}][n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$	14 64 15 65	$[1+(-1)^{\tau_3}][n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$	44 54 45 55	$n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}$ $+[(-1)^{\tau_2}+(-1)^{\tau_3}][n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$
$(n^2,m^2;n^3,m^3) = $ (odd,even;odd,odd)							
$(\sigma_2;\sigma_3) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$	
11 41 16 46	$[n^1\varepsilon_0^{(1)} + m^1\tilde{\varepsilon}_0^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$	51 61 56 66	$[1+(-1)^{\tau_2}][n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$	14 44 15 45	$[1+(-1)^{\tau_3}][n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$	54 64 55 65	$[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$ $+[(-1)^{\tau_2}+(-1)^{\tau_3}][n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$
$(n^2,m^2;n^3,m^3) = $ (even,odd;odd,odd)							
$(\sigma_2;\sigma_3) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$	
11 51 16 56	$[n^1\varepsilon_0^{(1)} + m^1\tilde{\varepsilon}_0^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$	41 61 46 66	$[1+(-1)^{\tau_2}][n^1\varepsilon_1^{(1)} + m^1\tilde{\varepsilon}_1^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$	14 54 15 55	$[1+(-1)^{\tau_3}][n^1\varepsilon_2^{(1)} + m^1\tilde{\varepsilon}_2^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$ $+(-1)^{\tau_2+\tau_3}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$	44 64 45 65	$[1+(-1)^{\tau_2+\tau_3}][n^1\varepsilon_3^{(1)} + m^1\tilde{\varepsilon}_3^{(1)}]$ $+(-1)^{\tau_2}[n^1\varepsilon_5^{(1)} + m^1\tilde{\varepsilon}_5^{(1)}]$ $+(-1)^{\tau_3}[n^1\varepsilon_4^{(1)} + m^1\tilde{\varepsilon}_4^{(1)}]$

Exceptional three-cycles from the $\mathbb{Z}_2^{(1)}$ sector of $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$ in dependence of wrapping numbers, Wilson lines and displacements

Table 52: Relation among wrapping numbers (n^i, m^i), discrete displacements $\vec{\sigma}$ and Wilson lines $\vec{\tau}$, $\mathbb{Z}_2^{(1)}$ fixed points and exceptional three-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_6$. The exceptional contribution to a fractional cycle will be multiplied by an over-all factor $(-1)^{\tau_0^{(1)}}/4$, where $\tau_0^{(1)} \in \{0,1\}$ parameterises the $\mathbb{Z}_2^{(1)}$ eigenvalue.

Exceptional three-cycles from the $\mathbb{Z}_2^{(2)}$ sector of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$			
f.p.$^{(2)}$	orbit	f.p.$^{(2)}$	orbit
$(n^3, m^3) = $ (odd,odd)			
$\sigma_3 = 0$		1	
$k1$ $k6$	$(-1)^{\tau_3}[-(n^2+m^2)\varepsilon_k^{(2)} + n^2\tilde{\varepsilon}_k^{(2)}]$	$k4$ $k5$	$[n^2 + (-1)^{\tau_3} m^2]\varepsilon_k^{(2)} + [(1-(-1)^{\tau_3})m^2 - (-1)^{\tau_3} n^2]\tilde{\varepsilon}_k^{(2)}$

Table 53: Relation among torus wrapping numbers (n^i, m^i), discrete displacements $\vec{\sigma}$ and Wilson lines $\vec{\tau}$, $\mathbb{Z}_2^{(2)}$ fixed points and exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$. For brevity of the table, we keep $k \in \{1 \ldots 4\}$ as a free label on the first two-torus $T^2_{(1)}$. The contribution to a fractional cycle is multiplied by the $\mathbb{Z}_2^{(2)}$ eigenvalue times a normalisation factor, $(-1)^{\tau_0^{(2)}}/4$.

Exceptional three-cycles from the $\mathbb{Z}_2^{(3)}$ sector of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$			
f.p.$^{(3)}$	orbit	f.p.$^{(3)}$	orbit
$(n^2, m^2) = (\text{odd,odd})$			
$\sigma_2 = 0$		1	
$k1$ $k6$	$(-1)^{\tau_2}[-(n^3+m^3)\varepsilon_k^{(3)} + n^3\tilde{\varepsilon}_k^{(3)}]$	$k4$ $k5$	$[n^3 + (-1)^{\tau_2} m^3]\varepsilon_k^{(3)} + [(1-(-1)^{\tau_2})m^3 - (-1)^{\tau_2} n^3]\tilde{\varepsilon}_k^{(3)}$
$(n^2, m^2) = (\text{odd,even})$			
$\sigma_2 = 0$		1	
$k1$ $k4$	$(-1)^{\tau_2}[n^3\varepsilon_k^{(3)} + m^3\tilde{\varepsilon}_k^{(3)}]$	$k5$ $k6$	$[m^3 - (-1)^{\tau_2}(n^3+m^3)]\varepsilon_k^{(3)}$ $+[-(n^3+m^3) + (-1)^{\tau_2} n^3]\tilde{\varepsilon}_k^{(3)}$
$(n^2, m^2) = (\text{even,odd})$			
$\sigma_2 = 0$		1	
$k1$ $k5$	$(-1)^{\tau_2}[m^3\varepsilon_k^{(3)} - (n^3+m^3)\tilde{\varepsilon}_k^{(3)}]$	$k4$ $k6$	$[n^3 - (-1)^{\tau_2}(n^3+m^3)]\varepsilon_k^{(3)} + [m^3 + (-1)^{\tau_2} n^3]\tilde{\varepsilon}_k^{(3)}$

Table 54: Relation among wrapping numbers (n^i, m^i), discrete displacements $\vec{\sigma}$ and Wilson lines $\vec{\tau}$, $\mathbb{Z}_2^{(3)}$ fixed points and exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$. For brevity of the table, we keep $k \in \{1 \ldots 4\}$ as a free label on the first two-torus $T^2_{(1)}$. The contribution to a fractional cycle is multiplied by the $\mathbb{Z}_2^{(3)}$ eigenvalue times a normalisation factor, $(-1)^{\tau_0^{(3)}}/4$.

f.p.$^{(i)}$	orbit	f.p.$^{(i)}$	orbit	f.p.$^{(i)}$	orbit	f.p.$^{(i)}$	orbit
Exceptional three-cycles, wrapping numbers, Wilson lines and displacements on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ with discrete torsion, Part I							
$(n^j, m^j; n^k, m^k) = $ (odd,odd;odd,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}\left[m^i\varepsilon_1^{(i)} - (n^i+m^i)\tilde\varepsilon_1^{(i)}\right]$	41	$((1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i)\varepsilon_1^{(i)}$	14	$((1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i)\varepsilon_2^{(i)}$	44	$((1-(-1)^{\tau_j+\tau_k})n^i - (-1)^{\tau_j+\tau_k}m^i)\varepsilon_3^{(i)}$
61	$+(-1)^{\tau_k}\left[m^i\varepsilon_2^{(i)} - (n^i+m^i)\tilde\varepsilon_2^{(i)}\right]$	51	$+(m^i+(-1)^{\tau_j}n^i)\tilde\varepsilon_1^{(i)}$	64	$+(m^i+(-1)^{\tau_k}n^i)\tilde\varepsilon_2^{(i)}$	54	$+(m^i+(-1)^{\tau_j+\tau_k}n^i)\tilde\varepsilon_3^{(i)}$
16	$+(-1)^{\tau_j+\tau_k}\left[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde\varepsilon_3^{(i)}\right]$	46	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde\varepsilon_4^{(i)}]$	15	$+(-1)^{\tau_j}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde\varepsilon_4^{(i)}]$	45	$+(-1)^{\tau_k}[n^i\varepsilon_4^{(i)} + m^i\tilde\varepsilon_4^{(i)}]$
66		56	$+(-1)^{\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde\varepsilon_5^{(i)}]$	65	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde\varepsilon_5^{(i)}]$	55	$+(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde\varepsilon_5^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (odd,even;odd,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[n^i\varepsilon_1^{(i)} + m^i\tilde\varepsilon_1^{(i)}]$	51	$[-n^i + ((-1)^{\tau_j} - 1)m^i]\varepsilon_1^{(i)}$	14	$((1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i)\varepsilon_2^{(i)}$	54	$[-n^i + ((-1)^{\tau_j+\tau_k} - 1)m^i]\varepsilon_5^{(i)}$
41	$+(-1)^{\tau_k}[m^i\varepsilon_2^{(i)} - (n^i+m^i)\tilde\varepsilon_2^{(i)}]$	61	$+[(1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i]\tilde\varepsilon_1^{(i)}$	44	$+(m^i+(-1)^{\tau_k}n^i)\tilde\varepsilon_2^{(i)}$	64	$+[(1-(-1)^{\tau_j+\tau_k})n^i - (-1)^{\tau_j+\tau_k}m^i]\tilde\varepsilon_5^{(i)}$
16	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde\varepsilon_5^{(i)}]$	56	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde\varepsilon_4^{(i)}]$	15	$+(-1)^{\tau_j}[n^i\varepsilon_3^{(i)} + m^i\tilde\varepsilon_3^{(i)}]$	55	$+(-1)^{\tau_j}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde\varepsilon_4^{(i)}]$
46		66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde\varepsilon_3^{(i)}]$	45	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_4^{(i)} + m^i\tilde\varepsilon_4^{(i)}]$	65	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde\varepsilon_3^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (even,odd;odd,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_1^{(i)} + n^i\tilde\varepsilon_1^{(i)}]$	41	$[n^i + (-1)^{\tau_j}m^i]\varepsilon_1^{(i)}$	14	$((1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i)\varepsilon_2^{(i)}$	44	$n^i\varepsilon_3^{(i)} + m^i\tilde\varepsilon_3^{(i)}$
51	$+(-1)^{\tau_k}[m^i\varepsilon_2^{(i)} - (n^i+m^i)\tilde\varepsilon_2^{(i)}]$	61	$+[-(-1)^{\tau_j}n^i + (1-(-1)^{\tau_j})n^i]\tilde\varepsilon_1^{(i)}$	54	$+(m^i+(-1)^{\tau_k}n^i)\tilde\varepsilon_2^{(i)}$	64	$+((-1)^{\tau_j}m^i + (-1)^{\tau_k}n^i)\varepsilon_4^{(i)}$
16	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde\varepsilon_4^{(i)}]$	46	$+(-1)^{\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde\varepsilon_5^{(i)}]$	15	$+(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde\varepsilon_5^{(i)}]$	45	$+(-(-1)^{\tau_j}n^i + ((-1)^{\tau_k} - (-1)^{\tau_j})m^i)\tilde\varepsilon_4^{(i)}$
56		66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde\varepsilon_3^{(i)}]$	55	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde\varepsilon_3^{(i)}]$	65	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde\varepsilon_5^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (odd,odd;odd,even)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[m^i\varepsilon_1^{(i)} - (n^i+m^i)\tilde\varepsilon_1^{(i)}]$	41	$((1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i)\varepsilon_1^{(i)}$	15	$[-n^i + ((-1)^{\tau_k} - 1)m^i]\varepsilon_2^{(i)}$	45	$[(1-(-1)^{\tau_j+\tau_k})n^i - (-1)^{\tau_j+\tau_k}m^i]\varepsilon_4^{(i)}$
61	$+(-1)^{\tau_k}[n^i\varepsilon_2^{(i)} + m^i\tilde\varepsilon_2^{(i)}]$	51	$+(m^i+(-1)^{\tau_j}n^i)\tilde\varepsilon_1^{(i)}$	65	$+[(1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i]\tilde\varepsilon_2^{(i)}$	55	$+[(-1)^{\tau_j+\tau_k}n^i + m^i]\tilde\varepsilon_4^{(i)}$
14	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde\varepsilon_4^{(i)}]$	44	$+(-1)^{\tau_k}[n^i\varepsilon_3^{(i)} + m^i\tilde\varepsilon_3^{(i)}]$	16	$+(-1)^{\tau_j}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde\varepsilon_5^{(i)}]$	46	$+(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde\varepsilon_3^{(i)}]$
64		54	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde\varepsilon_5^{(i)}]$	66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde\varepsilon_3^{(i)}]$	56	$+(-1)^{\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde\varepsilon_5^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (odd,odd;even,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[m^i\varepsilon_1^{(i)} - (n^i+m^i)\tilde\varepsilon_1^{(i)}]$	41	$((1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i)\varepsilon_1^{(i)}$	14	$[n^i + (-1)^{\tau_k}m^i]\varepsilon_2^{(i)}$	44	$n^i\varepsilon_3^{(i)} + m^i\tilde\varepsilon_3^{(i)}$
61	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_2^{(i)} + n^i\tilde\varepsilon_2^{(i)}]$	51	$+(m^i+(-1)^{\tau_j}n^i)\tilde\varepsilon_1^{(i)}$	64	$+[-(-1)^{\tau_k}n^i + (1-(-1)^{\tau_k})m^i]\tilde\varepsilon_2^{(i)}$	54	$+([(-1)^{\tau_k} - (-1)^{\tau_j}]n^i - (-1)^{\tau_j}m^i)\varepsilon_5^{(i)}$
15	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde\varepsilon_5^{(i)}]$	45	$+(-1)^{\tau_k}[n^i\varepsilon_4^{(i)} + m^i\tilde\varepsilon_4^{(i)}]$	16	$+(-1)^{\tau_j}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde\varepsilon_4^{(i)}]$	46	$+((-1)^{\tau_j}n^i + (-1)^{\tau_k}m^i)\tilde\varepsilon_5^{(i)}$
65		55	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde\varepsilon_3^{(i)}]$	66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde\varepsilon_3^{(i)}]$	56	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde\varepsilon_4^{(i)}]$

Table 55: Relation among wrapping numbers, $\mathbb{Z}_2^{(i)}$ fixed points and exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ with discrete torsion, Part I. The contribution to a fractional cycle is multiplied by the $\mathbb{Z}_2^{(i)}$ eigenvalue times a normalisation factor, $(-1)^{\tau_0^{(i)}}/4$.

f.p.	orbit	f.p.	orbit	f.p.	orbit	f.p.	orbit
Exceptional three-cycles, wrapping numbers, Wilson lines and displacements on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ with discrete torsion, Part II							
$(n^j, m^j; n^k, m^k) = $ (odd,even;odd,even)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[n^i\varepsilon_1^{(i)} + m^i\tilde{\varepsilon}_1^{(i)}]$	51	$[-n^i + ((-1)^{\tau_j} - 1)m^i]\varepsilon_1^{(i)}$	15	$[-n^i + ((-1)^{\tau_k} - 1)m^i]\varepsilon_2^{(i)}$	55	$[-n^i + ((-1)^{\tau_j+\tau_k} - 1)m^i]\varepsilon_3^{(i)}$
41	$+(-1)^{\tau_k}[n^i\varepsilon_2^{(i)} + m^i\tilde{\varepsilon}_2^{(i)}]$	61	$+[(1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i]\tilde{\varepsilon}_1^{(i)}$	45	$+[(1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i]\tilde{\varepsilon}_2^{(i)}$	65	$+[(1-(-1)^{\tau_j+\tau_k})n^i - (-1)^{\tau_j+\tau_k}m^i]\tilde{\varepsilon}_3^{(i)}$
14	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_3^{(i)} + m^i\tilde{\varepsilon}_3^{(i)}]$	54	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde{\varepsilon}_5^{(i)}]$	16	$+(-1)^{\tau_j}[n^i\varepsilon_4^{(i)} + m^i\tilde{\varepsilon}_4^{(i)}]$	56	$+(-1)^{\tau_j}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde{\varepsilon}_5^{(i)}]$
44		64	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde{\varepsilon}_4^{(i)}]$	46	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde{\varepsilon}_5^{(i)}]$	66	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde{\varepsilon}_4^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (odd,even;even,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[n^i\varepsilon_1^{(i)} + m^i\tilde{\varepsilon}_1^{(i)}]$	51	$[-n^i + ((-1)^{\tau_j} - 1)m^i]\varepsilon_1^{(i)}$	14	$[n^i + (-1)^{\tau_k}m^i]\varepsilon_2^{(i)}$	54	$[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde{\varepsilon}_5^{(i)}]$
41	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_2^{(i)} + n^i\tilde{\varepsilon}_2^{(i)}]$	61	$+[(1-(-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i]\tilde{\varepsilon}_1^{(i)}$	44	$+[-(-1)^{\tau_k}n^i + (1-(-1)^{\tau_k})m^i]\tilde{\varepsilon}_2^{(i)}$	64	$+[-(-1)^{\tau_k}n^i + ((-1)^{\tau_j} - (-1)^{\tau_k})m^i]\varepsilon_4^{(i)}$
15	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_4^{(i)} + m^i\tilde{\varepsilon}_4^{(i)}]$	55	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde{\varepsilon}_3^{(i)}]$	16	$+(-1)^{\tau_j}[n^i\varepsilon_3^{(i)} + m^i\tilde{\varepsilon}_3^{(i)}]$	56	$+[((-1)^{\tau_k} - (-1)^{\tau_j})n^i - (-1)^{\tau_j}m^i]\tilde{\varepsilon}_4^{(i)}$
45		65	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde{\varepsilon}_5^{(i)}]$	46	$+(-1)^{\tau_j+\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde{\varepsilon}_5^{(i)}]$	66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde{\varepsilon}_3^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (even,odd;odd,even)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_1^{(i)} + n^i\tilde{\varepsilon}_1^{(i)}]$	41	$[n^i + (-1)^{\tau_j}m^i]\varepsilon_1^{(i)}$	15	$[-n^i + ((-1)^{\tau_k} - 1)m^i]\varepsilon_2^{(i)}$	45	$[n^i\varepsilon_4^{(i)} + m^i\tilde{\varepsilon}_4^{(i)}]$
51	$+(-1)^{\tau_k}[n^i\varepsilon_2^{(i)} + m^i\tilde{\varepsilon}_2^{(i)}]$	61	$+[-(-1)^{\tau_j}n^i + (1-(-1)^{\tau_j})m^i]\tilde{\varepsilon}_1^{(i)}$	55	$+[(1-(-1)^{\tau_k})n^i - (-1)^{\tau_k}m^i]\tilde{\varepsilon}_2^{(i)}$	65	$+[(-1)^{\tau_k}n^i + (-1)^{\tau_j}m^i]\varepsilon_5^{(i)}$
14	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde{\varepsilon}_5^{(i)}]$	44	$+(-1)^{\tau_k}[n^i\varepsilon_3^{(i)} + m^i\tilde{\varepsilon}_3^{(i)}]$	16	$+(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde{\varepsilon}_3^{(i)}]$	46	$+[-(-1)^{\tau_j}n^i + ((-1)^{\tau_k} - (-1)^{\tau_j})m^i]\tilde{\varepsilon}_5^{(i)}$
54		64	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde{\varepsilon}_4^{(i)}]$	56	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde{\varepsilon}_4^{(i)}]$	66	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_3^{(i)} - (n^i+m^i)\tilde{\varepsilon}_3^{(i)}]$
$(n^j, m^j; n^k, m^k) = $ (even,odd;even,odd)							
	$(\sigma_j;\sigma_k) = (0;0)$		$(1;0)$		$(0;1)$		$(1;1)$
11	$(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_1^{(i)} + n^i\tilde{\varepsilon}_1^{(i)}]$	41	$[n^i + (-1)^{\tau_j}m^i]\varepsilon_1^{(i)}$	14	$[n^i + (-1)^{\tau_k}m^i]\varepsilon_2^{(i)}$	44	$[n^i + (-1)^{\tau_j+\tau_k}m^i]\varepsilon_3^{(i)}$
51	$+(-1)^{\tau_k}[-(n^i+m^i)\varepsilon_2^{(i)} + n^i\tilde{\varepsilon}_2^{(i)}]$	61	$+[-(-1)^{\tau_j}n^i + (1-(-1)^{\tau_j})m^i]\tilde{\varepsilon}_1^{(i)}$	54	$+[-(-1)^{\tau_k}n^i + (1-(-1)^{\tau_k})m^i]\tilde{\varepsilon}_2^{(i)}$	64	$+[-(-1)^{\tau_j+\tau_k}n^i + (1-(-1)^{\tau_j+\tau_k})m^i]\tilde{\varepsilon}_3^{(i)}$
15	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_3^{(i)} + n^i\tilde{\varepsilon}_3^{(i)}]$	45	$+(-1)^{\tau_k}[n^i\varepsilon_4^{(i)} + m^i\tilde{\varepsilon}_4^{(i)}]$	16	$+(-1)^{\tau_j}[-(n^i+m^i)\varepsilon_5^{(i)} + n^i\tilde{\varepsilon}_5^{(i)}]$	46	$+(-1)^{\tau_j}[m^i\varepsilon_4^{(i)} - (n^i+m^i)\tilde{\varepsilon}_4^{(i)}]$
55		65	$+(-1)^{\tau_j+\tau_k}[m^i\varepsilon_5^{(i)} - (n^i+m^i)\tilde{\varepsilon}_5^{(i)}]$	56	$+(-1)^{\tau_j+\tau_k}[-(n^i+m^i)\varepsilon_4^{(i)} + n^i\tilde{\varepsilon}_4^{(i)}]$	66	$+(-1)^{\tau_k}[n^i\varepsilon_5^{(i)} + m^i\tilde{\varepsilon}_5^{(i)}]$

Table 56: Relation among wrapping numbers, $\mathbb{Z}_2^{(i)}$ fixed points and exceptional three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ with discrete torsion, Part II.

Hodge numbers per twist sector on $T^4/\mathbb{Z}_N$

$T^4/$	lattice Hodge nr.	Untwisted	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	total
$\mathbb{Z}_2$	$SU(2)^4$		$(\frac{1}{2},-\frac{1}{2})$			
	h_{11}	4	16			20
$\mathbb{Z}_3$	$SU(3)^2$		$(\frac{1}{3},-\frac{1}{3})$			
	h_{11}	2	18			$2+18$
$\mathbb{Z}_4$	$SO(5)^2$		$(\frac{1}{4},-\frac{1}{4})$	$(\frac{1}{2},-\frac{1}{2})$		
	h_{11}	2	8	10		$12+8$
$\mathbb{Z}_6$	$SU(3)^2$		$(\frac{1}{6},-\frac{1}{6})$	$(\frac{1}{3},-\frac{1}{3})$	$(\frac{1}{2},-\frac{1}{2})$	
	h_{11}	2	2	10	6	$8+12$

Hodge numbers per twist sector on $T^6/\mathbb{Z}_N$

$T^6/$	lattice Hodge numbers	Untwisted	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	total
$\mathbb{Z}_3$	$SU(3)^3$		$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$			
	h_{11}	9	27			36
	h_{21}	0	0			0
$\mathbb{Z}_4$	$SU(2)^2\times SO(5)^2$		$(-\frac{1}{2},\frac{1}{4},\frac{1}{4})$	$(0,\frac{1}{2},-\frac{1}{2})$		
	h_{11}	5	16	10		31
	h_{21}	1	0	6		7
$\mathbb{Z}_6$	$SU(3)^3$		$(-\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	
	h_{11}	5	3	15	6	29
	h_{21}	0	0	0	5	5
$\mathbb{Z}_6'$	$SU(2)^2\times SU(3)^2$		$(-\frac{1}{2},\frac{1}{3},\frac{1}{6})$	$(0,-\frac{1}{3},\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
	h_{11}	3	12	12	8	35
	h_{21}	1	0	6	4	$5+6$

Table 57: Distribution of the Hodge numbers of toroidal orbifold limits of K3 (left) and of Calabi-Yau three-folds (right) per twist sector. Intersecting D6-branes (D7-branes) in Type IIA (IIB) orientifolds can wrap three-cycles (two-cycles) from the untwisted and $\mathbb{Z}_2$ twisted sectors only whose counting is highlighted in blue. Since $b_3 = 2h_{21} + 2$ for Calabi-Yau three-folds ($b_2 = h_{11} + 2$ for K3), there are two more three-cycles (two-cycles) from the untwisted sector on which D6-branes (D7-branes) can be wrapped.

Hodge numbers (h_{11}^+, h_{11}^-) per twist sector for IIA on $T^6/(\mathbb{Z}_N\times\Omega\mathcal{R})$

$T^6/$ Hodge nr.	Untw.	$\vec{w}$	$2\vec{w}$	$3\vec{w}$	total
$\mathbb{Z}_3$		$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$			
(h_{11}^+,h_{11}^-)	$(3,6)$	**AAA** $(13,14)$ **AAB** $(12,15)$ **ABB** $(9,18)$ **BBB** $(0,27)$			**AAA** $(16,20)$ **AAB** $(15,21)$ **ABB** $(12,24)$ **BBB** $(3,33)$
$\mathbb{Z}_4$		$(-\frac{1}{2},\frac{1}{4},\frac{1}{4})$	$(0,\frac{1}{2},-\frac{1}{2})$		
(h_{11}^+,h_{11}^-)	$(1,4)$	$(8\,b,16-8b)$	**a/bAA**, **a/bBB** $(0,10)$ **a/bAB** $(1,9)$		**a/bAA**, **a/bBB** $(1+8\,b,30-8\,b)$ **a/bAB** $(2+8\,b,29-8\,b)$
$\mathbb{Z}_6$		$(-\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$(0,\frac{1}{2},-\frac{1}{2})$	
(h_{11}^+,h_{11}^-)	$(1,4)$	**AAA**, **AAB**, **ABB** $(1,2)$ **BBB** $(0,3)$	**AAA**, **ABB** $(5,10)$ **AAB** $(6,9)$ **BBB** $(0,15)$	$(1,5)$	**AAA**, **ABB** $(8,21)$ **AAB** $(9,20)$ **BBB** $(2,27)$
$\mathbb{Z}_6'$		$(-\frac{1}{2},\frac{1}{3},\frac{1}{6})$	$(0,-\frac{1}{3},\frac{1}{3})$	$(\frac{1}{2},0,-\frac{1}{2})$	
(h_{11}^+,h_{11}^-)	$(0,3)$	**a/bAA**, **a/bAB** $(4+2b,8-2b)$ **a/bBA**, **a/bBB** $(6b,12-6b)$	**a/bAA**, **a/bAB** $(4,8)$ **a/bBA**, **a/bBB** $(0,12)$	$(4b,8-4b)$	**a/bAA**, **a/bAB** $(8+6b,27-6b)$ **a/bBA**, **a/bBB** $(10b,35-10b)$

Table 58: The number of Kähler moduli and Abelian vectors in $T^6/\mathbb{Z}_N$ orientifolds depends on the orientation of the factorisable lattices.

Two-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_3$		
$T^4/\mathbb{Z}_3$	$T^6/\mathbb{Z}_6'$	fixed point counting
$\sum_{k=0}^2 \omega(\pi_{35})$	$\emptyset$	
$\sum_{k=0}^2 \omega(\pi_{36})$	$\emptyset$	
$3\,\pi_{34}$	$6\,\pi_{34}$	
$3\,\pi_{56}$	$6\,\pi_{56}$	
$\emptyset$	$6\,\pi_{12}$	
$d_{l,l\in\{1\ldots9\}}^{\mathbb{C}^2/\mathbb{Z}_3(k),k\in\{1,2\}}$	$2\,d_{l,l\in\{1,2,3\}}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k),k\in\{1,2\}}$ $2\,\big(d_{l=3+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)} + d_{l=6+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)}\big)_{i\in\{1,2,3\}}^{k\in\{1,2\}}$	$l \in \{1\ldots3\} \Leftrightarrow \{(1i)\}$ $l \in \{4\ldots6\} \Leftrightarrow \{(2i)\}$ $l \in \{7\ldots9\} \Leftrightarrow \{(3i)\}$ on $T_2 \times T_3$ with $i = 1, 2, 3$
$\emptyset$	$d_{l,l\in\{1\ldots12\}}^{\mathbb{C}^3/\mathbb{Z}_6'}$	$l \in \{1\ldots12\} \Leftrightarrow$ $\{(i1k)\}$ on $T_1 \times T_2 \times T_3$ with $i = 1, 2, 3, 4$ and $k = 1, 2, 3$
$\emptyset$	$2\,e_{l,l\in\{1\ldots8\}}^{(2)}$	$l \in \{1\ldots4\} \Leftrightarrow \{(i1)\}$ $l \in \{5\ldots8\} \Leftrightarrow \{(i4+i5+i6)\}$ on $T_1 \times T_3$ with $i = 1, 2, 3, 4$

Three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_3$			
$T^4/\mathbb{Z}_3$		$T^6/\mathbb{Z}_6'$	fixed point counting
$\sum_{k=0}^2 \omega(\pi_{35})$	$\xrightarrow{2\pi_1\otimes}$ $\xrightarrow{2\pi_2\otimes}$	$\hat{\rho}_1$ $\hat{\rho}_3$	
$\sum_{k=0}^2 \omega(\pi_{35})$	$\xrightarrow{2\pi_1\otimes}$ $\xrightarrow{2\pi_2\otimes}$	$\hat{\rho}_2$ $\hat{\rho}_4$	
$3\,\pi_{34}$		$\emptyset$	
$3\,\pi_{56}$		$\emptyset$	
$d_{l,l\in\{1\ldots9\}}^{\mathbb{C}^2/\mathbb{Z}_3(k),k\in\{1,2\}}$	$\xrightarrow{2\pi_1\otimes}$ $\xrightarrow{2\pi_2\otimes}$	$2\,\pi_1 \otimes \big(d_{l=3+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)} - d_{l=6+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)}\big)_{i\in\{1,2,3\}}^{k\in\{1,2\}}$ $2\,\pi_2 \otimes \big(d_{l=3+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)} - d_{l=6+i}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k)}\big)_{i\in\{1,2,3\}}^{k\in\{1,2\}}$	$l \in \{4\ldots6\} \Leftrightarrow \{(2i)\}$ $l \in \{7\ldots9\} \Leftrightarrow \{(3i)\}$ on $T_2 \times T_3$ with $i = 1, 2, 3$
$\emptyset$		$\hat{\varepsilon}_{k,k\in\{1\ldots4\}}$	$k \in \{1\ldots4\} \Leftrightarrow$ Orbit of $\pi_1 \otimes \{(i4)\}$ with $(i4)$ on $T_1 \times T_3$
$\emptyset$		$\hat{\tilde{\varepsilon}}_{k,k\in\{1\ldots4\}}$	$k \in \{1\ldots4\} \Leftrightarrow$ Orbit of $\pi_2 \otimes \{(i4)\}$ with $(i4)$ on $T_1 \times T_3$

Table 59: Two and three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_3$ and their origin from the $T^4/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_6'$ sub-sectors of the product orbifold, where $T^4 = T^2_{(2)} \times T^2_{(3)}$.

Two-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$		
$T^4/\mathbb{Z}_2$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$	
	$\eta = 1$	$\eta = -1$
$2\,\pi_{35}$	$\emptyset$	
$2\,\pi_{46}$	$\emptyset$	
$2\,\pi_{36}$	$\emptyset$	
$2\,\pi_{45}$	$\emptyset$	
$2\,\pi_{34}$	$4\,\pi_{34}$	
$2\,\pi_{56}$	$4\,\pi_{56}$	
$\emptyset$	$4\,\pi_{12}$	
$e_{l,l\in\{1\ldots16\}}$	$2\,e^{(1)}_{l,l\in\{1\ldots16\}}$	$\emptyset$
$\emptyset$	$2\,e^{(2)}_{l,l\in\{1\ldots16\}}$	$\emptyset$
$\emptyset$	$2\,e^{(3)}_{l,l\in\{1\ldots16\}}$	$\emptyset$

Three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$			
$T^4/\mathbb{Z}_2$		$T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$	
		$\eta = 1$	$\eta = -1$
$2\,\pi_{35}$	$\xrightarrow{2\pi_1\otimes}$	$4\,\pi_{135}$	
	$\xrightarrow{2\pi_2\otimes}$	$4\,\pi_{235}$	
$2\,\pi_{46}$	$\xrightarrow{2\pi_1\otimes}$	$4\,\pi_{146}$	
	$\xrightarrow{2\pi_2\otimes}$	$4\,\pi_{246}$	
$2\,\pi_{36}$	$\xrightarrow{2\pi_1\otimes}$	$4\,\pi_{136}$	
	$\xrightarrow{2\pi_2\otimes}$	$4\,\pi_{236}$	
$2\,\pi_{45}$	$\xrightarrow{2\pi_1\otimes}$	$4\,\pi_{145}$	
	$\xrightarrow{2\pi_2\otimes}$	$4\,\pi_{245}$	
$2\,\pi_{34}$	$\xrightarrow{2\pi_1\otimes}$	$\emptyset$	
	$\xrightarrow{2\pi_2\otimes}$	$\emptyset$	
$2\,\pi_{56}$	$\xrightarrow{2\pi_1\otimes}$	$\emptyset$	
	$\xrightarrow{2\pi_2\otimes}$	$\emptyset$	
e_l	$\xrightarrow{2\pi_1\otimes}$	$\emptyset$	$2\,e^{(1)}_{l,l\in\{1\ldots16\}} \otimes \pi_1$
	$\xrightarrow{2\pi_2\otimes}$	$\emptyset$	$2\,e^{(1)}_{l,l\in\{1\ldots16\}} \otimes \pi_2$
$\emptyset$		$\emptyset$	$2\,e^{(2)}_{l,l\in\{1\ldots16\}} \otimes \pi_3$
		$\emptyset$	$2\,e^{(2)}_{l,l\in\{1\ldots16\}} \otimes \pi_4$
$\emptyset$		$\emptyset$	$2\,e^{(3)}_{l,l\in\{1\ldots16\}} \otimes \pi_5$
		$\emptyset$	$2\,e^{(3)}_{l,l\in\{1\ldots16\}} \otimes \pi_6$

Table 60: Two and three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and their origin from the $T^4/\mathbb{Z}_2$ sub-sector with $T^4 = T^2_{(2)} \times T^2_{(3)}$ and $\mathbb{Z}_2 = \mathbb{Z}_2^{(1)}$ for the case without ($\eta = 1$) and with ($\eta = -1$) discrete torsion.

Two-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$				
$T^4/\mathbb{Z}_4$	$T^6/\mathbb{Z}_4$	$T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ $\eta=1$	$T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ $\eta=-1$	fixed point counting
$\emptyset$	$4\,\pi_{12}\xrightarrow{\times 2}$	$8\,\pi_{12}$		
$4\,\pi_{34}\xrightarrow{\times 2}$		$8\,\pi_{34}$		
$4\,\pi_{56}\xrightarrow{\times 2}$		$8\,\pi_{56}$		
$2(\pi_{35}-\pi_{46})$	$2(\pi_{35}+\pi_{46})$	$\emptyset$		
$2(\pi_{36}+\pi_{45})$	$2(\pi_{36}-\pi_{45})$	$\emptyset$		
$d^{\mathbb{C}^2/\mathbb{Z}_4(k),k\in\{0,1,2\}}_{l,l\in\{1\dots4\}}\xrightarrow{\times 2}$	$e^{(1)}_{l,l\in\{1\dots4\}}\xrightarrow{\times 2}$	$2\,d^{\mathbb{C}^4/\mathbb{Z}_4^{(1)}(k),k\in\{0,1,2\}}_{l,l\in\{1\dots4\}}$	$\left\{\begin{array}{c} 2\,e^{(1)}_{l,l\in\{1\dots4\}} \\ \equiv 2\,d^{\mathbb{C}^4/\mathbb{Z}_4^{(1)}(0)}_{l,l\in\{1\dots4\}}\end{array}\right.$	$l\in\{1\dots4\}\Leftrightarrow$ $\{(11),(13),(31),(33)\}$ on $T_2\times T_3$
$e_{l,l\in\{5\dots10\}}\xrightarrow{\times 2}$		$2\,e^{(1)}_{l,l\in\{5\dots10\}}$		$l\in\{5\dots10\}\Leftrightarrow$ $\{(12+14),(32+34),(21+41),$ $(23+43),(22+44),(24+42)\}$ on $T_2\times T_3$
$\emptyset$	$d^{\mathbb{C}^6/\mathbb{Z}_4}_{l,l\in\{1\dots16\}}\xrightarrow{\times 2}$	$2\,d^{\mathbb{C}^6/\mathbb{Z}_4}_{l,l\in\{1\dots16\}}$	$\emptyset$	$l\in\{1\dots16\}\Leftrightarrow$ $\{(i11),(i13),(i31),(i33)\}$ with $i=1,2,3,4$
$\emptyset$		$2\,e^{(2)}_{l,l\in\{1\dots12\}}$	$2\,e^{(2)}_{l,l\in\{9\dots12\}}$	$l\in\{1\dots8\}\Leftrightarrow\{(i1),(i3)\}$, $l\in\{9\dots12\}\Leftrightarrow\{(i2\pm i4)\}$, on $T_1\times T_3$ with $i=1,2,3,4$
$\emptyset$		$2\,e^{(3)}_{l,l\in\{1\dots12\}}$	$2\,e^{(3)}_{l,l\in\{9\dots12\}}$	$l\in\{1\dots8\}\Leftrightarrow\{(i1),(i3)\}$, $l\in\{9\dots12\}\Leftrightarrow\{(i2\pm i4)\}$, on $T_1\times T_2$ with $i=1,2,3,4$

Three-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$				
$T^4/\mathbb{Z}_4$	$T^6/\mathbb{Z}_4$	$T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ $\eta=1$	$T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ $\eta=-1$	fixed point counting
$2(\pi_{35}-\pi_{46})\xrightarrow{\otimes 2\pi_1}$	$\hat\rho_1\xrightarrow{\times 2}$	ρ_1		
$2(\pi_{36}+\pi_{45})\xrightarrow{\otimes 2\pi_2}$	$\hat\rho_2\xrightarrow{\times 2}$	ρ_2		
$-2(\pi_{35}-\pi_{46})\xrightarrow{\otimes 2\pi_2}$	$\hat\rho_3\xrightarrow{\times 2}$	ρ_3		
$2(\pi_{36}+\pi_{45})\xrightarrow{\otimes 2\pi_1}$	$\hat\rho_4\xrightarrow{\times 2}$	ρ_4		
$4\,\pi_{34}$	$\emptyset$	$\emptyset$		
$4\,\pi_{56}$	$\emptyset$	$\emptyset$		
$d^{\mathbb{C}^2/\mathbb{Z}_4(k),k\in\{0,1,2\}}_{l,l\in\{0\dots4\}}\xrightarrow{\otimes 2\pi_1}$	$\emptyset$	$\emptyset$	$\delta^{(k),k\in\{1,2\}}_{l,l\in\{1\dots4\}}$	$l\in\{1\dots4\}\Leftrightarrow$ $\{(11),(13),(31),(33)\}$ on $T_2\times T_3$
$d^{\mathbb{C}^2/\mathbb{Z}_4(k),k\in\{0,1,2\}}_{l,l\in\{0\dots4\}}\xrightarrow{\otimes 2\pi_2}$	$\emptyset$	$\emptyset$	$\tilde\delta^{(k),k\in\{1,2\}}_{l,l\in\{1\dots4\}}$	
$e_{l,l\in\{5\dots10\}}\xrightarrow{\otimes 2\pi_1}$	$\varepsilon^{(1)}_{l,l\in\{5\dots10\}}$	$\emptyset$		$l\in\{5\dots10\}\Leftrightarrow$ $\{(12\pm14),(32\pm34),(21\pm41),$ $(23\pm43),(22\pm44),(24\pm42)\}$ on $T_2\times T_3$
$\xrightarrow{\otimes 2\pi_2}$	$\tilde\varepsilon^{(1)}_{l,l\in\{5\dots10\}}$	$\emptyset$		

Table 61: Two and three-cycles on $T^6/\mathbb{Z}_2\times\mathbb{Z}_4$ and their origin from $T^4/\mathbb{Z}_4$ with $T^4=T^2_{(2)}\times T^2_{(3)}$ and $\mathbb{Z}_4=\mathbb{Z}_4^{(1)}$ and from $T^6/\mathbb{Z}_4$ for the case without ($\eta=1$) and with ($\eta=-1$) discrete torsion. .

Two-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$

$T^4/\mathbb{Z}_6$	$T^6/\mathbb{Z}_6'$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ $\eta = 1$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ $\eta = -1$	fixed point counting
$\emptyset$	$6\,\pi_{12} \xrightarrow{\times 2}$	$12\,\pi_{12}$		
$6\,\pi_{34} \xrightarrow{\times 2}$		$12\,\pi_{34}$		
$6\,\pi_{56} \xrightarrow{\times 2}$		$12\,\pi_{56}$		
$2\sum_{k=0}^{2}\omega(\pi_{35})$	$\emptyset$	$\emptyset$		
$2\sum_{k=0}^{2}\omega(\pi_{36})$	$\emptyset$	$\emptyset$		
$d_l^{\mathbb{C}^2/\mathbb{Z}_6(k),k\in\{0\ldots4\}}$	$d_l^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k),k\in\{1,2\}}$	$2d_l^{\mathbb{C}^2/\mathbb{Z}_6^{(1)}(k),k\in\{0\ldots4\}}$	$2d_l^{\mathbb{C}^2/\mathbb{Z}_6^{(1)}(k),k\in\{1,2\}}$	$l = (11)$ on $T_2 \times T_3$
$d_{l,l\in\{1\ldots4\}}^{\mathbb{C}^2/\mathbb{Z}_3(k),k\in\{1,2\}}$	$d_{l',l'\in\{1\ldots5\}}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k),k\in\{1,2\}}$	$2\,d_{l',l'\in\{1\ldots3\}}^{\mathbb{C}^2/\mathbb{Z}_3^{(1)}(k),k\in\{1,2\}}$		$l \in \{1\ldots4\} \Leftrightarrow$ $\{(12+13),(21+31),(22+33),(23+32)\}$, $l' \in \{1\ldots5\} \Leftrightarrow$ $\{(21),(31),(12+13),(22+23),(32+33)\}$, $l'' \in \{1\ldots3\} \Leftrightarrow$ $\{(12+13),(21+31),(22+33+23+32)\}$ on $T_2 \times T_3$
$e_{l,l\in\{1\ldots5\}} \xrightarrow{\times 2}$	$\emptyset$	$2\,e_{l,l\in\{1\ldots5\}}^{(1)}$	$\emptyset$	$l \in \{1\ldots5\} \Leftrightarrow$ $\{(14+15+16),(41+51+61),(44+56+65),$ $(45+54+66),(46+55+64)\}$ on $T_2 \times T_3$
$\emptyset$	$e_{l,l\in\{1\ldots8\}}^{(2)} \xrightarrow{\times 2}$	$2\,e_{l,l\in\{1\ldots8\}}^{(2)}$	$\emptyset$	$l \in \{1\ldots8\} \Leftrightarrow \{(i1),(i4+i5+i6)\}$ on $T_1 \times T_3$ with $i = 1,2,3,4$
$\emptyset$	$\emptyset$	$2\,e_{l,l\in\{1\ldots8\}}^{(3)}$	$\emptyset$	$l \in \{1\ldots8\} \Leftrightarrow \{(i1),(i4+i5+i6)\}$ on $T_1 \times T_2$ with $i = 1,2,3,4$
$\emptyset$	$d_{l,l\in\{1\ldots12\}}^{\mathbb{C}^3/\mathbb{Z}_6'} \xrightarrow{\times 2}$	$2\,d_{l',l'\in\{1\ldots8\}}^{\mathbb{C}^3/\mathbb{Z}_6'}$	$2\,d_{l'',l''\in\{1\ldots4\}}^{(\mathbb{C}^3/\mathbb{Z}_6')}$	$l \in \{1\ldots12\} \Leftrightarrow \{(i11),(i21),(i31)\}$ $l' \in \{1\ldots8\} \Leftrightarrow \{(i11),(i21+i31)\}$ $l'' \in \{1\ldots4\} \Leftrightarrow \{(i21-i31)\}$ on $T_1 \times T_2 \times T_3$ with $i = 1\ldots4$
$\emptyset$		$2\,d_{l',l'\in\{1\ldots8\}}^{\mathbb{C}^3/\tilde{\mathbb{Z}}_6'}$	$2\,d_{l'',l''\in\{1\ldots4\}}^{\mathbb{C}^3/\tilde{\mathbb{Z}}_6'}$	$l' \in \{1\ldots8\} \Leftrightarrow \{(i11),(i12+i13)\}$ $l'' \in \{1\ldots4\} \Leftrightarrow \{(i12-i13)\}$ on $T_1 \times T_2 \times T_3$ with $i = 1\ldots4$

Three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$

$T^4/\mathbb{Z}_6$	$T^6/\mathbb{Z}_6'$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ $\eta = 1$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ $\eta = -1$	fixed point counting
$2\sum_{k=0}^{2}\omega(\pi_{35}) \xrightarrow{2\pi_1\otimes}$	$\hat{\rho}_1 \xrightarrow{\times 2}$	ρ_1		
$2\sum_{k=0}^{2}\omega(\pi_{36}) \xrightarrow{2\pi_1\otimes}$	$\hat{\rho}_2 \xrightarrow{\times 2}$	ρ_2		
$2\sum_{k=0}^{2}\omega(\pi_{35}) \xrightarrow{2\pi_2\otimes}$	$\hat{\rho}_3 \xrightarrow{\times 2}$	ρ_3		
$2\sum_{k=0}^{2}\omega(\pi_{36}) \xrightarrow{2\pi_2\otimes}$	$\hat{\rho}_4 \xrightarrow{\times 2}$	ρ_4		
$6\,\pi_{34}$	$\emptyset$	$\emptyset$		
$6\,\pi_{56}$	$\emptyset$	$\emptyset$		
$e_{l,l\in\{1\ldots5\}} \xrightarrow{2\pi_1\otimes}$	$\emptyset$	$\emptyset$	$\varepsilon_{l,l\in\{1\ldots5\}}^{(1)}$	$l \in \{1\ldots5\} \Leftrightarrow \{(14+15+16),(41+51+61),$ $(44+56+65),(45+54+66),(46+55+64)\}$ on $T_2 \times T_3$
$\xrightarrow{2\pi_2\otimes}$	$\emptyset$	$\emptyset$	$\tilde{\varepsilon}_{l,l\in\{1\ldots5\}}^{(1)}$	
$d^{\mathbb{C}^2/\mathbb{Z}_6(k),k\in\{0\ldots4\}} \xrightarrow{2\pi_1\otimes}$	$\emptyset$	$\emptyset$	$\varepsilon_0^{(1)}, \delta^{(1),k\in\{3,4\}}$	(11) on $T_2 \times T_3$
$\xrightarrow{2\pi_2\otimes}$	$\emptyset$	$\emptyset$	$\tilde{\varepsilon}_0^{(1)}, \tilde{\delta}^{(1),k\in\{3,4\}}$	
$d_{l,l\in\{1\ldots4\}}^{\mathbb{C}^2/\mathbb{Z}_3(k),k\in\{1,2\}} \xrightarrow{2\pi_1\otimes}$	$\hat{\delta}_{l',l'\in\{1\ldots3\}}^{k,k\in\{1,2\}} \xrightarrow{\times 2}$	$\emptyset$	$\delta_{l''}^{(1),k\in\{1,2\}}$	$l \in \{1\ldots4\} \Leftrightarrow \{(12+13),(21+31),(22+33),(23+32)\}$ $l' \in \{1\ldots3\} \Leftrightarrow \{(i2+i3)\}$ with $i = 1,2,3$
$\xrightarrow{2\pi_2\otimes}$	$\hat{\tilde{\delta}}_{l',l'\in\{1\ldots3\}}^{k,k\in\{1,2\}} \xrightarrow{\times 2}$	$\emptyset$	$\tilde{\delta}_{l''}^{(1),k\in\{1,2\}}$	$l'' = (22+23+32+33)$ on $T_2 \times T_3$
$\emptyset$	$\hat{\varepsilon}_{l,l\in\{1\ldots4\}} \xrightarrow{\times 2}$	$\emptyset$	$\varepsilon_{l,l\in\{1\ldots4\}}^{(2)}$	$l \in \{1\ldots4\} \Leftrightarrow \{(i4+i5+i6)\}$
$\emptyset$	$\hat{\tilde{\varepsilon}}_{l,l\in\{1\ldots4\}} \xrightarrow{\times 2}$	$\emptyset$	$\tilde{\varepsilon}_{l,l\in\{1\ldots4\}}^{(2)}$	on $T_1 \times T_3$ with $i = 1\ldots4$
$\emptyset$		$\emptyset$	$\varepsilon_{l,l\in\{1\ldots4\}}^{(3)}$	$l \in \{1\ldots4\} \Leftrightarrow \{(i4+i5+i6)\}$
$\emptyset$		$\emptyset$	$\tilde{\varepsilon}_{l,l\in\{1\ldots4\}}^{(3)}$	on $T_1 \times T_2$ with $i = 1\ldots4$

Table 62: Two and three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6$ and their origin from $T^4/\mathbb{Z}_6$ with $T^4 = T^2_{(2)} \times T^2_{(3)}$ and $\mathbb{Z}_6 = \mathbb{Z}_6^{(1)}$ and from $T^6/\mathbb{Z}_6'$ for the case without $(\eta = 1)$ and with $(\eta = -1)$ discrete torsion.

Two-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$

$T^6/\mathbb{Z}_6$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ $\eta = 1$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ $\eta = -1$	fixed point counting
$6\,\pi_{12} \xrightarrow{\times 2}$	$12\,\pi_{12}$		
$6\,\pi_{34} \xrightarrow{\times 2}$	$12\,\pi_{34}$		
$6\pi_{56} \xrightarrow{\times 2}$	$12\,\pi_{56}$		
$\left.\begin{array}{c}\sum_{k=0}^{5}\theta^k(\pi_{35}) = \\ 4\pi_{35}+4\pi_{46}-2\pi_{45}-2\pi_{36}\end{array}\right\}$	$\emptyset$		
$\left.\begin{array}{c}\sum_{k=0}^{5}\theta^k(\pi_{36}) = \\ 2\pi_{35}+2\pi_{36}+2\pi_{46}-4\pi_{45}\end{array}\right\}$	$\emptyset$		
$d^{\mathbb{C}^3/\mathbb{Z}_6^{(1)}}_{l,l\in\{1\ldots 3\}}$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(1)}}_{l',l'\in\{1,2\}}$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(1)}}_{l''}$	$l \in \{1\ldots 3\} \Leftrightarrow \{(111),(211),(311)\}$ $l' \in \{1,2\} \Leftrightarrow \{(111),(211+311)\}$ $l'' = (211-311)$ on $T_1 \times T_2 \times T_3$
$d^{\mathbb{C}^3/\mathbb{Z}_3}_{l,l\in\{1\ldots 15\}}$	$d^{\mathbb{C}^3/\mathbb{Z}_3}_{l',l'\in\{1\ldots 9\}}$		$l,l' \in \{1,2,3\} \Leftrightarrow \{(111),(121+131),(112+113)\}$ $l \in \{4\ldots 9\} \Leftrightarrow \{(211),(221+231),$ $(212+213),(122+133),(222+233),(232+223)\}$ $l \in \{10\ldots 15\} \Leftrightarrow \{(311),(321+331),$ $(312+313),(132+123),(332+323),(322+333)\}$ $l' \in \{4\ldots 9\} \Leftrightarrow \{(l+[l+6])$ with $l \in 4\ldots 9\}$,
$e_{l,l\in\{1\ldots 6\}} \xrightarrow{\times 2}$	$2\,e^{(1)}_{l,l\in\{1\ldots 6\}}$	$\emptyset$	$l \in \{1\ldots 5\} \Leftrightarrow \{(11),(14+15+16),(41+51+61),$ $(44+55+66),(45+56+64),(46+54+65)\}$ on $T_2 \times T_3$
$\emptyset$	$2\,e^{(2)}_{l,l\in\{1\ldots 6\}}$	$\emptyset$	see $2\,e^{(1)}_l$ and permute $T_2 \leftrightarrow T_1$
$\emptyset$	$2\,e^{(3)}_{l,l\in\{1\ldots 6\}}$	$\emptyset$	$\ldots$ permute $T_3 \leftrightarrow T_1$
$\emptyset$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(2)}}_{l',l'\in\{1,2\}}$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(2)}}_{l''}$	see $2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(1)}}$ and permute $T_2 \leftrightarrow T_1$
$\emptyset$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(3)}}_{l',l'\in\{1,2\}}$	$2\,d^{\mathbb{C}^3/\mathbb{Z}_6^{(3)}}_{l''}$	$\ldots$ permute $T_3 \leftrightarrow T_1$

Three-cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$

$T^6/\mathbb{Z}_6$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ $\eta = 1$	$T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ $\eta = -1$	fixed point counting
$\hat{\rho}_1 \xrightarrow{\times 2}$	ρ_1		
$\hat{\rho}_2 \xrightarrow{\times 2}$	ρ_2		
$\hat{\varepsilon}_{l,l\in\{1\ldots 5\}} \xrightarrow{\times 2}$	$\emptyset$	$\varepsilon^{(1)}_{l,l\in\{1\ldots 5\}}$	$l \in \{1\ldots 5\} \Leftrightarrow \{(14+15+16),$ $(41+51+61),(44+55+66),$ $(45+56+64),(46+54+65)\}$ on $T_2 \times T_3$
$\hat{\tilde{\varepsilon}}^{(1)}_{l,l\in\{1\ldots 5\}} \xrightarrow{\times 2}$	$\emptyset$	$\tilde{\varepsilon}^{(1)}_{l,l\in\{1\ldots 5\}}$	
$\emptyset$	$\emptyset$	$\varepsilon^{(2)}_{l,l\in\{1\ldots 5\}}$	
$\emptyset$	$\emptyset$	$\tilde{\varepsilon}^{(2)}_{l,l\in\{1\ldots 5\}}$	$\ldots$ on $T_1 \times T_3$
$\emptyset$	$\emptyset$	$\varepsilon^{(3)}_{l,l\in\{1\ldots 5\}}$	
$\emptyset$	$\emptyset$	$\tilde{\varepsilon}^{(3)}_{l,l\in\{1\ldots 5\}}$	$\ldots$ on $T_1 \times T_2$

Table 63: Two and three cycles on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_6'$ and their origin from $T^6/\mathbb{Z}_6$ for the case without ($\eta = 1$) and with ($\eta = -1$) discrete torsion.

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